18 b) $\begin{cases} x' = x^2 - 2y^3 \\ y' = xy^2 + x^2y + \frac{1}{2}y^3 \end{cases}$

SOL.

INDEED (0,0) IS AN EQUILIBRIUM LET $V = ax^2 + by^2$

THEN

$$\dot{V} = V_x f_1 + V_y = 2ax(x^2 - 2y^3) + 2by(xy^2 + x^2y + \tfrac{1}{2}y^3)$$
$$= 2ax^3 - 4axy^3 + 2bxy^3 + 2bx^2y^2 + by^4$$
$$= 2ax^4 + by^4 + 2bx^2y^2 + 2xy^3(b - 2a) .$$

LET $a = 1, b = 2$. THEN $V = x^2 + 2by^2$ IS (+D)

AND $\dot{V} = 2x^4 + 4x^2y^2 + y^4$. WE CONSIDER

$$b^2 - 4ac = 16 - 4(2)(1) = 8 > 0$$

$\longrightarrow$ $\dot{V}$ IS (+D). THEREFORE, BY THM (0,0) IS (NS).

18c $\begin{cases} x' = y^3 - 2x \\ y' = x - y - xy^2 \end{cases}$

SOL.

INDEED (0,0) IS AN EQUILIBRIUM. LET $V = ax^2 + by^2$.

THEN

$$\dot{V} = 2ax(y^3 - 2x) + 2by(x - y - xy^2)$$
$$= 2axy^3 - 4ax^2 + 2byx - 2by^2 - 2by^3x$$
$$= -4ax^2 - 2by^2 + xy^3(2a - 2b).$$

LET $a = 1 = b$ THEN $V = x^2 + y^2$ IS (+D) AND

$\dot{V} = -(4x^2 + 2y^2)$ IS (−D). THEREFORE BY

THM (0,0) IS (US) & (AS).

20. $\begin{cases} x' = y + kx^3 \\ y' = -x + ky^5 \end{cases}$

SOL.

INDEED (0,0) IS AN EQUILIBRIUM, LET $V = ax^2 + by^2$

THEN

$$\dot{V} = 2ax(y + kx^3) + 2by(-x + ky^5)$$
$$= 2axy + 2akx^4 - 2byx + 2bky^6$$
$$= 2akx^4 + 2by^6 + xy(2a - 2b)$$

LET $a = b = 1$. THEN $V = x^2 + y^2$ IS (+D) AND $\dot{V} = 2k(x^4 + y^6)$.

IF $k > 0$, THEN $\dot{V}$ IS (+D), AND BY THM IMPLIES (0,0) IS (NS).

IF $k < 0$, THEN $\dot{V}$ IS (−D), AND BY THM IMPLIES (0,0) IS (US) & (AS).

IF $k = 0$, THEN $V(x(t), x'(t)) = V(x(0), x'(0))$ $\forall\, t \in [0, \infty)$.

THIS IMPLIES $\{(x(t), x'(t)) \mid t \in [0,\infty)\}$ IS IN A LEVEL CURVE $V(x, x') = c$

FOR SOME $c > 0$. ALL $V(x, x')$ ARE CLOSED

$\longrightarrow$ STABILITY $\Rightarrow$ (0,0) IS (S).

21 $\begin{cases} x' = y - x^3 y^2 \\ y' = -x^3 - y^3 x^2 \end{cases}$

a)

INDEED $(0,0)$ IS AN EQUILIBRIUM. DEFINE

$$A = \begin{bmatrix} -3x^2y^2 & 1 - x^3(2y) \\ -3x^2 - 2y^3x & -3y^2x^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

THEN $\lambda_1(A) = 0$ AND $\lambda_2(A) = 0$. SINCE 0 IS NOT IN THE JORDAN DIAGONALIZED BLOCK $(0,0)$ IS (NS).

b) INDEED $(0,0)$ IS AN EQUILIBRIUM. LET $V = ax^4 + by^2$

THEN

$$\dot{V} = 4ax^3(y - x^3y^2) + 2by(-x^3 - y^3x^2)$$
$$= 4ax^3y - 4ax^6y^2 - 2byx^3 - 2by^4x^2$$
$$= -4ax^6y^2 - 2by^4x^2 + x^3y(4a - 2b)$$

LET $a = 1$, $b = 2$. THEN $V = x^4 + 2y^2$ IS (+D) AND

$-4x^2y^2(x^4+y^2) = \dot{V} = -(4x^6y^2 + 4y^4x^2)$ IS (-SD). THEREFORE, BY THM

$(0,0)$ IS (AS). $\therefore$ (A) $(0,0)$ IS (US) (AS).

22 a) $\begin{cases} x' = x^3(y - b) \\ y' = y^3(x - a)^5 \end{cases}$ $\quad a, b > 0$

INDEED $(0,0)$ IS AN EQUILIBRIUM. LET $V = x^2 + y^2$

THEN

$$\dot{V} = 2x(x^3(y-b)) + 2y(y^3(x-a)^5)$$
$$= 2x^4(y-b) + 2y^4(x-a)^5$$

WE KNOW $2x^4 \geq 0$ $\forall x$ AND $2y^4 \geq 0$ $\forall y$.

WTS $(y-b) < 0$ AND $(x-a)^5 < 0$ $\forall x, y$ SMALL AND $a, b > 0$.

FIX ANY $a, b > 0$ AND LET $(x,y) \in \mathbb{R}^2$. THEN $\exists r > 0$

S.t. $(a,b) \notin B((0,0), r)$ WHERE $B((0,0), r)$ IS THE OPEN BALL

IN $\mathbb{R}$ CENTERED @ $(0,0)$ W/ RADIUS r. ~~THEREFORE~~ THEN, FOR

SUFFICIENTLY SMALL (x,y) WE HAVE $|x| < a$ AND $|y| < b$.

THUS, $x - a < 0$ AND $y - b < 0$. THEREFORE, $V = x^2 + y^2$ IS (+D)

AND $\dot{V}$ IS (-D). BY THM $(0,0)$ IS (US) AND (AS).

FACT $B((0,0), r) \neq \emptyset$.

22. (1) $\begin{cases} x' = x^3(y-b) \\ y' = y^3(x-a)^5 \end{cases}$

INDEED (a,b) IS AN EQUILIBRIUM OF (1). LET $u = y-b$ AND $w = x-a$. THEN $x = w+a$, $y = u+b$, $y' = u'$, AND $x' = w'$.
DEFINE (2) $\begin{cases} u' = (u+b)^3(w)^5 \\ w' = (w+a)^3(u) \end{cases}$

INDEED $(0,0)$ IS AN EQUILIBRIUM OF (2). LET $V = uw$
THEN

$$\dot{V} = V_u f_1 + V_w f_2 = w(u+b)^3 w^5 + u(w+a)^3 u$$
$$= w^6(u+b)^3 + u^2(w+a)^3$$

AS SHOWN $|u| < b$ AND $|w| < a$ BY REPLACING x WITH w AND y WITH u FROM PREVIOUS ARGUMENT. THEREFORE, FOR ANY FIXED $a,b>0$ AND (x,y) IN A NBHD OF $(0,0)$ WE HAVE $u+b>0$ AND $w+a>0$. THUS, $\dot{V}$ IS (+D).
WE NOTE $V = uw$ IS (I.D). THEREFORE, BY THM $(0,0)$ OF (2) IS (NS) $\Longrightarrow$ (a,b) OF (1) IS (NS).

23. $\begin{cases} x' = y + 2y^3 \\ y' = -x - 2x^3 \end{cases}$

SOL

INDEED $(0,0)$ IS AN EQUILIBRIUM. LET $V = ax^2 + bx^4 + cy^2 + dy^4$
THEN

$$\dot{V} = 2ax(y+2y^3) + 4bx^3(y+2y^3) + 2cy(-x-2x^3) + 4dy^3(-x-2x^3)$$
$$= xy(2a-2c) + xy^3(4a-4d) + x^3y(4b-4c) + x^3y^3(8b-8d)$$

LET $a=b=c=d=1$. THEN $V = x^2+x^4+y^2+y^4$ IS (+D).
AND $\dot{V} = 0$ IS (−SD). THEREFORE, BY THM $(0,0)$ OF (A) IS (S) $\Longrightarrow$ $(0,0)$ IS (US).

24 $\begin{cases} x' = y \\ y' = z \\ z' = -y \end{cases}$

INDEED $(0,0,0)$ IS AN EQUILIBRIUM. LET $V = (x+z)^2 + y^2 + z^2$
THEN

$$\dot{V} = 2(x+z)^2(y-y) + 2yz - 2yz = 0$$

SO, V IS (+D) AND $\dot{V}$ IS (−SD). THEREFORE, $(0,0)$ IS (S)
$\Longrightarrow$ $(0,0,0)$ IS (US).

24. $\begin{cases} x' = -x - y + z + x^3 \\ y' = x - 2y + 2z + xz \\ z' = x + 2y + z + xy \end{cases}$

INDEED $(0,0,0)$ IS AN EQUILIBRIUM. LET $V = -x^2 - y^2 + 2z^2$
THEN

$$\dot{V} = -2x(-x - y + z + x^3) - 2y(x - 2y + 2z + xz) + 4z(x + 2y + z + xy)$$
$$= 2x^2 + 2y^2 + 2z^2 .$$

SO, V IS (ID) AND $\dot{V}$ IS (+D). THEREFORE $(0,0,0)$ IS (NS).

26 (1) $u'' - (|u| + |u'| - 1)u' + u|u| = 0$

SOL

LET $x = u$ AND $y = u'$ THEN (2) $\begin{cases} x' = y \\ y' = y|x| + y|y| - y - x|u| \end{cases}$

HERE WE USE THE ENERGY FUNCTION $V = \dfrac{y^2}{2} + \displaystyle\int_0^x g(s)\,ds$

WHERE $g(s) = s|s|$. NOTICE $sg(s) = s^2|s| > 0 \ \forall s$
$\Longrightarrow \int_0^x g(s)\,ds > 0$ FOR SMALL x. SO, V IS (+D).
NOW THEREFORE

$$\dot{V} = V_x f_1 + V_y f_2 = g(x)y + y(y|x| + y|y| - y - x|x|)$$
$$= yx|x| + y^2|x| + y^2|y| - y^2 - yx|x|$$
$$= y^2(|x| + |y| - 1)$$

LET $z = |x| + |y|$. THEN z IS ALWAYS POSITIVE. FURTHER AS z GETS SUFFICIENTLY SMALL ENOUGH SAY $z < 1$ WE HAVE $z - 1 < 0$. SINCE $y^2 \geq 0 \ \forall y$, AND $z - 1 \leq 0$ FOR z IN ANY NBHD OF ZERO WE HAVE $\dot{V}$ IS (−D).
THEREFORE, BY THM $u = 0$ OF (1) IS (US) AND (AS).

27. CONSIDER (1) $u'' + u' + au^3 + f(u) = 0$ $a > 0$ $f: \mathbb{R} \to \mathbb{R}$ AND CONT. AND $f(u)/u^3 \to 0$ AS $u \to 0$. THEN $u = 0$ IS A SOL. OF THE EQ AND (AS).

PROOF

LET $x = u$ AND $y = u'$, THEN (2) $\begin{cases} x' = y \\ y' = -y - ax^3 - f(x) \end{cases}$

WE FIRST WANT TO SHOW $f(0) = 0$. SINCE $f(x)$ IS CONTINUOUS $\forall x \in \mathbb{R}$ WE KNOW $\lim_{x \to 0} f(x) = f(0)$.

WE ARE GIVEN $\lim_{x \to 0} \frac{f(x)}{x^3} = 0 \Rightarrow \frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} x^3} = 0$

$\Rightarrow \lim_{x \to 0} f(x) = 0$. THUS, $f(0) = 0$.

$\therefore$ (0,0) IS AN EQUILIBRIUM OF (2).

$\therefore$ $u = 0$ IS A SOL. OF (1). //

WTS $u = 0$ OF (1) IS (AS).

WE USE THE ENERGY FUNCTION $V = \frac{y^2}{2} + \int_0^x g(s)\,ds$

WHERE $g(s) = au^3 + f(u)$. WE THEN EVALUATE TO GET

$$\int_0^x g(s)\,ds = \int_0^x ax^3 + f(s)\,ds = \frac{ax^4}{4} + \int_0^x f(s)\,ds.$$

SINCE f IS CONTINUOUS AND $\lim_{x \to 0} f(x) = 0 = f(0)$

WE HAVE $f(x) < 0 \iff x < 0$
$f(x) > 0 \iff x > 0$?

$\Rightarrow x f(x) > 0 \Rightarrow \int_0^x f(s)\,ds \geq 0 \Rightarrow V$ IS (+D).

NOW $\dot{V} = V_x f_1 + V_y f_2 = (ax^3 - f(x))y + y(-y - ax^3 - f(x)) = -y^2$

$\Rightarrow \dot{V}$ IS (−SD). ~~$\therefore$ (0,0) OF (2) IS (AS)~~ ?

$\therefore$ $u = 0$ OF (1) IS (AS).