27. CONSIDER (1) $u'' + u' + au^3 + f(u) = 0$ $\quad a>0 \quad f: \mathbb{R} \to \mathbb{R}$ AND CONT. AND $f(u)/u^3 \to 0$ AS $u \to 0$. THEN $u=0$ IS A SOL. OF THE EQ AND (AS).

PROOF

LET $x = u$ AND $y = u'$, THEN (2) $\begin{cases} x' = y \\ y' = -y - ax^3 - f(x) \end{cases}$

WE FIRST WANT TO SHOW $f(0) = 0$. SINCE $f(x)$ IS CONTINUOUS $\forall x \in \mathbb{R}$ WE KNOW $\lim_{x \to 0} f(x) = f(0)$.

WE ARE GIVEN $\lim_{x \to 0} \frac{f(x)}{x^3} = 0 \Rightarrow \dfrac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} x^3} = 0$

$\Rightarrow \lim_{x \to 0} f(x) = 0$. THUS, $f(0) = 0$.

$\therefore$ (0,0) IS AN EQUILIBRIUM OF (2).
$\therefore$ $u=0$ IS A SOL. OF (1). //

WTS $u=0$ OF (1) IS (AS).
WE USE THE ENERGY FUNCTION $V = \frac{y^2}{2} + \int_0^x g(s)\,ds$

WHERE $g(s) = au^3 + f(u)$. WE THEN EVALUATE TO GET

$$\int_0^x g(s)\,ds = \int_0^x ax^3 + f(s)\,ds = \frac{ax^4}{4} + \int_0^x f(s)\,ds.$$

SINCE f IS CONTINUOUS AND $\lim_{x \to 0} f(x) = 0 = f(0)$

WE HAVE $f(x) < 0 \Longleftrightarrow x < 0$
$f(x) > 0 \Longleftrightarrow x > 0$?
$\Rightarrow x f(x) > 0 \Rightarrow \int_0^x f(s)\,ds \geq 0 \Rightarrow V$ IS (+D).

NOW $\dot{V} = V_x f_1 + V_y f_2 = (ax^3 - f(x))y + y(-y - ax^3 - f(x)) = -y^2$
$\Rightarrow \dot{V}$ IS (−SD). ~~$\therefore$ (0,0) OF (2) IS (AS)~~ ?
$\therefore$ $u=0$ OF (1) IS (AS).