18 b) $\begin{cases} x' = x^2 - 2y^3 \\ y' = xy^2 + x^2y + \frac{1}{2}y^3 \end{cases}$

SOL.

INDEED (0,0) IS AN EQUILIBRIUM LET $V = ax^2 + by^2$

THEN

$$\dot{V} = V_x f_1 + V_y = 2ax(x^2 - 2y^3) + 2by(xy^2 + x^2y + \tfrac{1}{2}y^3)$$
$$= 2ax^3 - 4axy^3 + 2bxy^3 + 2bx^2y^2 + by^4$$
$$= 2ax^4 + by^4 + 2bx^2y^2 + 2xy^3(b - 2a) \quad .$$

LET $a = 1, b = 2$. THEN $V = x^2 + 2by^2$ IS (+D)

AND $\dot{V} = 2x^4 + 4x^2y^2 + y^4$. WE CONSIDER

$$b^2 - 4ac = 16 - 4(2)(1) = 8 > 0$$

$\longrightarrow$ $\dot{V}$ IS (+D). THEREFORE, BY THM (0,0) IS (NS).

18c $\begin{cases} x' = y^3 - 2x \\ y' = x - y - xy^2 \end{cases}$

SOL.

INDEED (0,0) IS AN EQUILIBRIUM. LET $V = ax^2 + by^2$.

THEN

$$\dot{V} = 2ax(y^3 - 2x) + 2by(x - y - xy^2)$$
$$= 2axy^3 - 4ax^2 + 2byx - 2by^2 - 2by^3x$$
$$= -4ax^2 - 2by^2 + xy^3(2a - 2b).$$

LET $a = 1 = b$. THEN $V = x^2 + y^2$ IS (+D) AND

$\dot{V} = -(4x^2 + 2y^2)$ IS (−D). THEREFORE BY

THM (0,0) IS (US) & (AS).

20. $\begin{cases} x' = y + kx^3 \\ y' = -x + ky^5 \end{cases}$

SOL.

INDEED (0,0) IS AN EQUILIBRIUM, LET $V = ax^2 + by^2$

THEN

$$\dot{V} = 2ax(y + kx^3) + 2by(-x + ky^5)$$
$$= 2axy + 2akx^4 - 2byx + 2bky^6$$
$$= 2akx^4 + 2by^6 + xy(2a - 2b)$$

LET $a = b = 1$. THEN $V = x^2 + y^2$ IS (+D) AND $\dot{V} = 2k(x^4 + y^6)$.

IF $k > 0$, THEN $\dot{V}$ IS (+D), AND BY THM IMPLIES (0,0) IS (NS).

IF $k < 0$, THEN $\dot{V}$ IS (−D), AND BY THM IMPLIES (0,0) IS (US) & (AS).

IF $k = 0$, THEN $V(x(t), x'(t)) = V(x(0), x'(0))$ $\forall t \in [0, \infty)$.

THIS IMPLIES $\{(x(t), x'(t)) \mid t \in [0, \infty)\}$ IS IN A LEVEL CURVE $V(x, x') = c$

FOR SOME $c > 0$. ALL $V(x, x')$ ARE CLOSED

$\longrightarrow$ STABILITY $\Longrightarrow$ (0,0) IS (S).