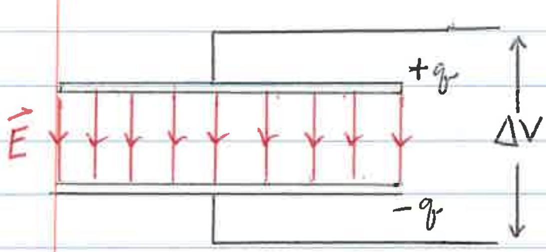
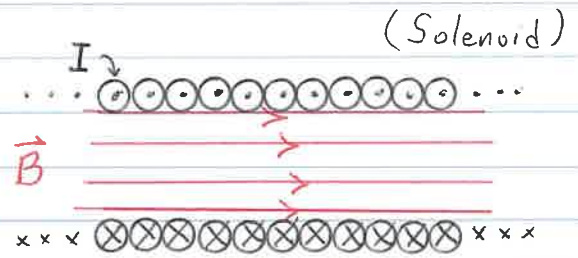
Inductors & Inductance





Set up $\vec{E}$-field with a capacitor (⊣⊢).

Set up a $\vec{B}$-field with an inductor (−ꝏꝏ−).

Analogy −ꝏꝏ− : $\vec{B}$ :: ⊣⊢ : $\vec{E}$

Also, resistors in a DC circuit restrict I, whereas inductors in a DC circuit restrict changes in I.

Just as $C \equiv \left|\frac{q}{\Delta V}\right|$ (capacitance), define "inductance", $\boxed{L \equiv \frac{N\Phi}{I}}$

'q' on plates gives rise to 'ΔV' between the plates.

'I' established in windings produces a flux Φ_B in each winding (N = # of windings)

Consider a solenoid (like above) with length ℓ and cross-sec. area A containing N loops. Calculate its 'L'.

$$L = \frac{N\Phi}{I} = \frac{NBA}{I}.$$

measured in $\frac{T \cdot m^2}{A}$

For solenoid, recall $B^* = \mu_0 n I$ where $n = N/\ell$ = # loops/unit length

$$L = \frac{N \mu_0 n \cancel{I} A}{\cancel{I}}$$

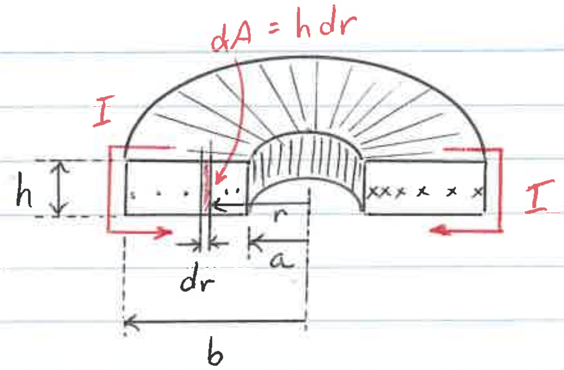
$$= \boxed{\frac{N^2 \mu_0 A}{\ell} \text{ or } \mu_0 n^2 A \ell}$$

Define 1 "Henry" = 1 $\frac{T \cdot m^2}{A}$ as the S.I unit of L.

Note: Like capacitance, inductance depends ONLY on the geometry of the inductor.

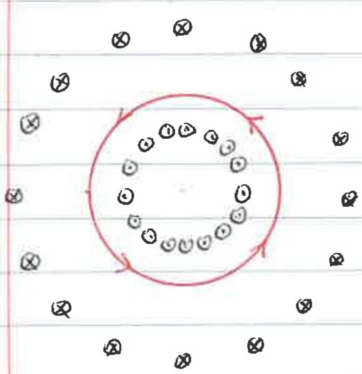
* Found using Ampere's law.

Inductance of a (rectangular) toriod



From Ampere's law,

$$B = \frac{\mu_0 N I}{2\pi r} \quad \text{inside}$$



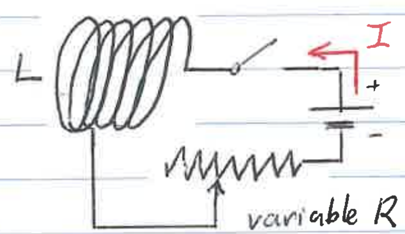
$$d\Phi_B = \vec{B}\cdot d\vec{A} = B\,dA$$

$$\Phi_B = \int B\,dA = \int_a^b \left(\frac{\mu_0 I N}{2\pi r}\right)(h\,dr)$$

$$= \frac{\mu_0 I N h}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{N\Phi_B}{I} = \left(\frac{N}{\cancel{I}}\right)\left(\frac{\mu_0 \cancel{I} N h}{2\pi}\right)\ln\left(\frac{b}{a}\right) = \boxed{\frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)}$$

Self Inductance :



some time after switch is closed, current reaches steady-state.

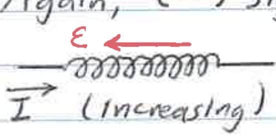
$$\mathcal{E} = -\frac{d}{dt}(N\Phi_B)$$

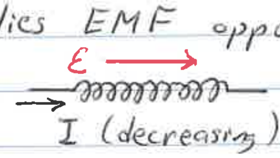
(Faraday's law)

$$L = \frac{N\Phi_B}{I} \rightarrow N\Phi_B = LI \quad \therefore \quad \mathcal{E} = -\frac{d}{dt}(LI)$$

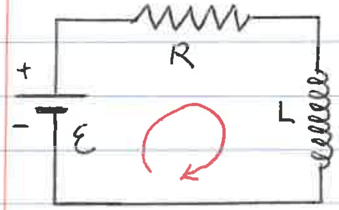
Voltage across inductor : $\boxed{\mathcal{E} = -L\dfrac{dI}{dt}}$

Again, (−) sign implies EMF opposes change:





Energy in Inductors (& RL Circuits)



Kirchoff's Loop Rule: $\sum_{loop} \Delta V = 0$

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

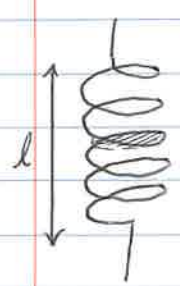
$$\mathcal{E} I - I^2 R - L I \frac{dI}{dt} = 0$$

Power supplied by battery.

Thermal dissipation rate in resistor.

Rate at which potential energy is <u>stored</u> in inductor

U_B = (magnetic) potential energy



N loops

Area, A

$$\frac{dU_B}{\cancel{dt}} = L I \frac{dI}{\cancel{dt}}$$

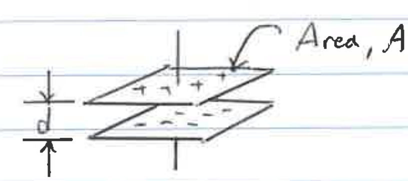
$$U_B = \int_0^{U_B} dU_B = L\int_0^I I\,dI = \boxed{\frac{1}{2} L I^2}$$

For solenoid: $L = \dfrac{\mu_0 N^2 A}{\ell}$, and $B = \mu_0 \left(\dfrac{N}{\ell}\right) I$

$\therefore\ U_B = \dfrac{1}{2}\dfrac{1}{\mu_0} B^2 (A\ell)$ Volume of inductor (solenoid).

Define magnetic energy density: $u_B = \dfrac{U_B}{\text{unit volume}}$

Then $\boxed{u_B = \dfrac{B^2}{2\mu_0}}$



Note the similarity to

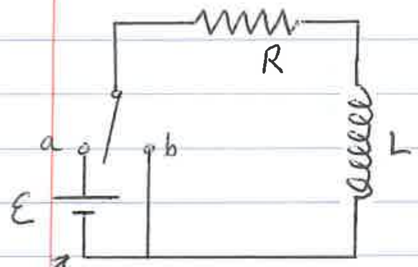
$$U_E = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)(Ed)^2$$

$$= \frac{1}{2}\varepsilon_0 E^2 (Ad)$$

Volume of capacitor

So, $u_E = \frac{1}{2}\varepsilon_0 E^2$

RL Circuits



Recall RC circuits:

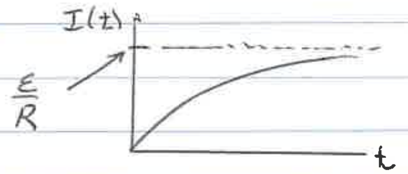
$$q(t) = C\mathcal{E}(1 - e^{-t/RC}), \text{ charging}$$

$$q(t) = Q_0 e^{-t/RC}, \text{ discharging}$$

An analogous situation occurs for the rise and fall of currents in an RL circuit.

Switch moved to position 'a':

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \longrightarrow I(t) = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$



(rise in current)

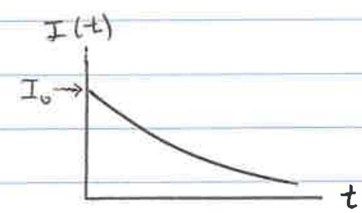
Inductor opposes increase in I.

After long time in position 'a', $\frac{dI}{dt} \to 0$ (and inductor behaves like a wire).

Switch moved to position 'b':

$$L\frac{dI}{dt} + IR = 0 \longrightarrow I(t) = I_0 e^{-Rt/L} \quad \left(\text{drop in } I\right)$$

$I_0 = \frac{\mathcal{E}}{R}$ if switch was in position 'a' for a long time.



Inductor opposes decrease in I.

$\frac{L}{R} \equiv$ inductive time constant

$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})$

$I(t) = I_0 e^{-t/\tau}$

Note that if the current through the inductor is initially zero, the inductor behaves like a "break".

When the current through the inductor is steady (unchanging), the inductor behaves like a wire. ($\Delta V_L = 0$).

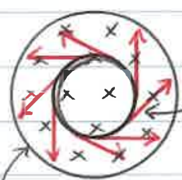
Faraday's Law: $\mathcal{E} = -\frac{d\Phi_B}{dt}$ The EMF induced can set up a current.

$\therefore$ An $\vec{E}$-field must be present.

Punchline: A changing $\vec{B}$-field produces an $\vec{E}$-field.

Now we need to generalize Faraday's law to cases the field within the loop is non-uniform or not $\perp$ to the plane of the loop.

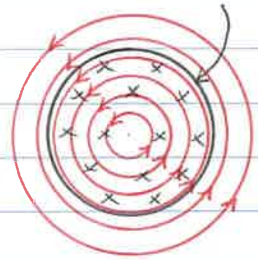
Recall magnetic flux through a loop: $\Phi_B = \int \vec{B} \cdot d\vec{A}$



Consider a region on spatially uniform $\vec{B}$ whose magnitude is increasing.

A conducting ring will set up a CCW current.

$\vec{B}$ increasing keeping area constant



Remove conducting ring a place a (+) test charge q_0 a distance 'r' from the center.

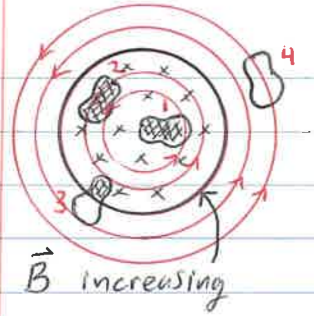
The q_0 will move in a circle because of the $\vec{E}$-field induced by the changing B-field.

The work done by the induced $\vec{E}$-field in one revolution is $q_0\mathcal{E} = \oint \vec{F} \cdot d\vec{s} = q_0 \oint \vec{E} \cdot d\vec{s}$

$$\therefore \mathcal{E} = \oint \vec{E} \cdot d\vec{s} = E \oint ds = E(2\pi r)$$

This is true for any closed path drawn in the $\vec{B}$-field.

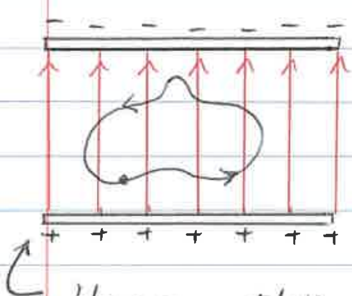
Faraday's law generalized: $\boxed{\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}}$



Consider the four closed loops shown. All have the same area. The induced EMF is the same for loops 1 & 2. The EMF for loop 3 is smaller (due to smaller changing magnetic flux through it). The induced EMF in loop 4 is ZERO.

In other words the rankings of $\frac{d\Phi_B}{dt}$ through the four closed paths are:

$$① = ② > ③ > ④ (=0)$$



Note similarity to electric potential:

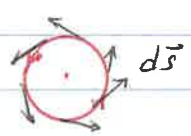
$$\Delta V = V_f - V_i = \frac{W_{if}}{q} = -\int_i^f \vec{E}\cdot d\vec{s}$$

In a closed path $\Delta V = 0 \rightarrow \oint \vec{E}\cdot d\vec{s} = 0$

However, this is for $\vec{E}$-field produced by static charges.

But in the situation described on the previous page,

$$\oint \vec{E}\cdot d\vec{s} = E(2\pi r)$$ for $\vec{E}$-field induced by changing $\vec{B}$-fields.



Since $\oint \vec{E}\cdot d\vec{s} \neq 0$ for the closed path, the concept of electric potential becomes meaningless for $\vec{E}$-fields induced by changing B-fields.

(If $W_{field} = 5\,J$ in one revolution, the potential at the same point would be 5 V greater which is impossible!)