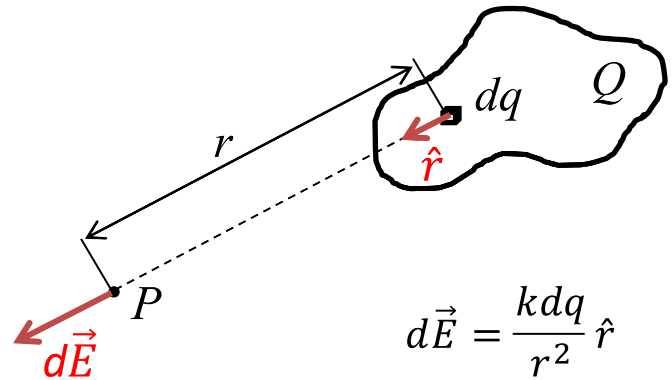
Magnetic Fields due to Currents

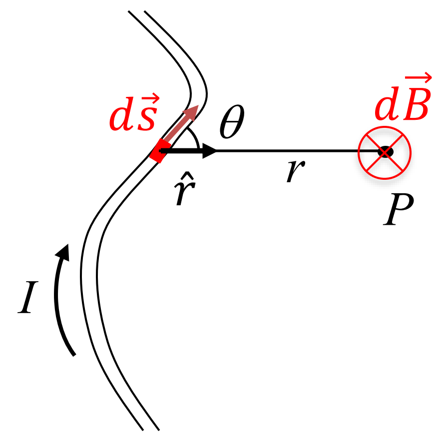
Biot-Savart Law & Ampere's Law

Defining Magnetic Field

Just as we defined the $\vec{E}$-field at a point P due to a distribution of charge, we use a similar procedure to define a $\vec{B}$-field.



Consider arbitrary wire carrying a steady current. An expression for the magnetic field at a point P was discovered by Jean Baptiste Biot and Felix Savart in 1820. The $\vec{B}$ -field point P from each segment of the wire depends on several factors…



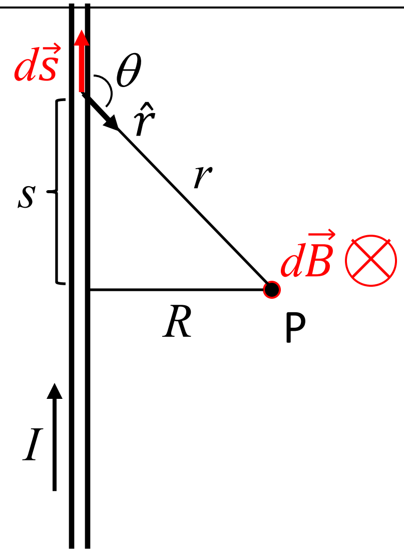
Experiments show…

1. $d\vec{B} \perp d\vec{s}$ and $d\vec{B} \perp \hat{r}$
2. $d\vec{B} \propto \frac{1}{r^2}$, where r is the radial distance from the $d\vec{s}$ to P.
3. $d\vec{B} \propto I$ and $d\vec{B} \propto$ length of $d\vec{s}$
4. $d\vec{B} \propto \sin\theta$, where q is the angle between $d\vec{s}$ and $\hat{r}$.

Summarizing these result compactly: $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \hat{r}}{r^2}$

This relationship is known as the Biot-Savart law.

Biot-Savart Law Example #1



Calculate the $\vec{B}$-field a distance R from a long straight wire carrying a steady current *I*.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}\times\hat{r}}{r^2},$$

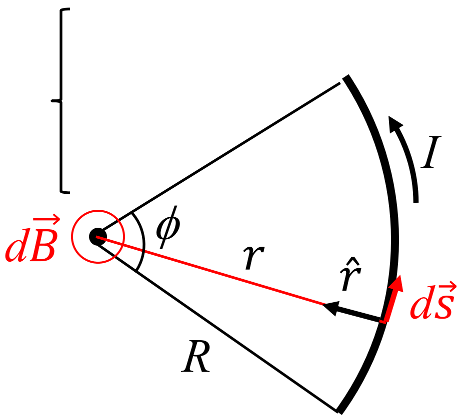
where $d\vec{s} \times \hat{r} = (ds)(1)(\sin\theta)$ in this case.

Note: $r^2 = s^2 + R^2$ and $\sin(180° - \theta) = \sin\theta = \frac{R}{(s^2 + R^2)^{1/2}}$.

So, $\left|\frac{d\vec{s}\times\hat{r}}{r^2}\right| = \frac{R\,ds}{(s^2+R^2)^{3/2}}$, directed into screen at point P.

$\therefore\ |\vec{B}| = \int dB = \frac{\mu_0 IR}{4\pi}\int_{-\infty}^{+\infty}\frac{ds}{(s^2+R^2)^{3/2}}$ or $B = \frac{\mu_0 I}{2\pi R}$.

Biot-Savart Law Example #2



Calculate the $\vec{B}$-field the center of a circular arc spanning an angle ϕ and of radius R carrying a steady current I.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}\times\hat{r}}{r^2},$$

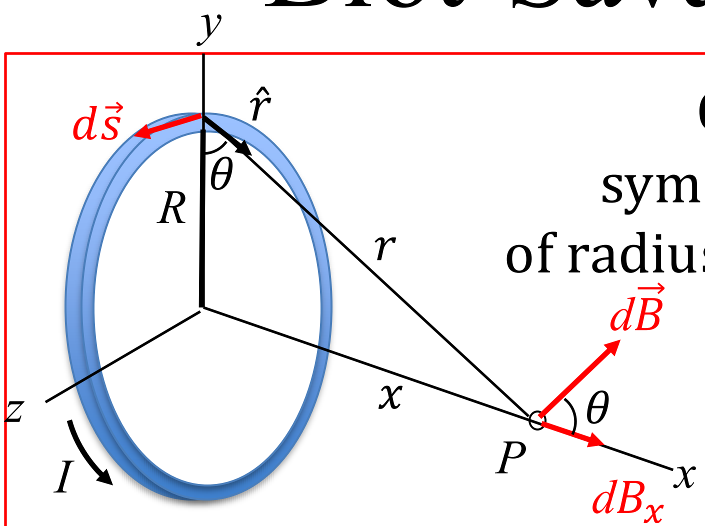
where $d\vec{s}\times\hat{r} = (ds)(1)(\sin 90°) = ds$ in this case.

Note: $r = \mathrm{R} =$ constant and $ds = Rd\phi$. So, $\left|\frac{d\vec{s}\times\hat{r}}{r^2}\right| = \frac{Rd\phi}{R^2} = \frac{d\phi}{R}$ and is directed out of the screen by the RHR.

$$\therefore \left|\vec{B}\right| = \int dB = \frac{\mu_0 I}{4\pi R}\int_0^{\phi} d\phi = \frac{\mu_0 I}{4\pi R} \quad \text{or} \quad B = \frac{\mu_0 I\phi}{4\pi R}.$$

(Note that ϕ must be in radian units in this expression!)

Biot-Savart Law Example #3



Calculate the $\vec{B}$-field a distance x along the symmetry axis from the center of a ring of radius R carrying a steady current I.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2},$$

where $d\vec{s}\times\hat{r} = (ds)(1)(\sin\theta)$ in this case.

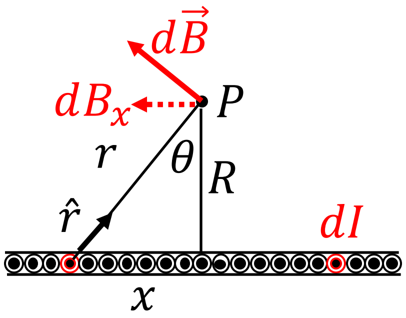
By symmetry, $B_y = B_z = 0$. $dB_x = dB\cos\theta$, where $\cos\theta = \frac{R}{r}$

Note: $r = (x^2 + R^2)^{1/2}$, so that $\left|\frac{d\vec{s}\times\hat{r}}{r^2}\right| \cos\theta = \frac{R\,ds}{(x^2+R^2)^{3/2}}$ = (constant)(ds).

$$\therefore\ |\vec{B}| = \int dB = \frac{\mu_0 I R}{4\pi(x^2+R^2)^{3/2}}\oint ds \quad \text{or} \quad \vec{B} = \frac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}\,\hat{\imath}\,.$$

Biot-Savart Law Example #4

Calculate the $\vec{B}$-field a distance R from a thin infinite sheet that carries a current dI per length dx.



$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}\times\hat{r}}{r^2}$, where $d\vec{s}\times\hat{r} = (ds)(1)(\sin 90°)$ in this case.

By symmetry, $B_y = 0$. $dB_x = dB\sin\theta$, where $\sin\theta = \frac{R}{r}$

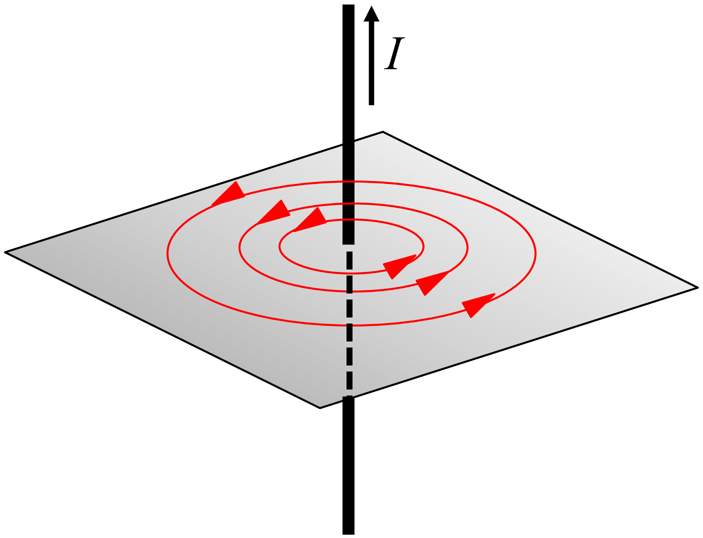
Note: $r = (x^2 + R^2)^{½}$, so that $\left|\frac{d\vec{s}\times\hat{r}}{r^2}\right|\sin\theta = \frac{R\,dx}{(x^2+R^2)^{3/2}}$

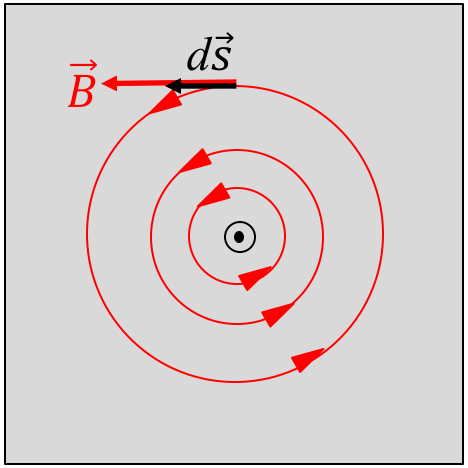
$\therefore \left|\vec{B}\right| = \int_{-\infty}^{+\infty} dB_x = 2\int_0^{+\infty}\frac{\mu_0 R\,dI}{2\pi(x^2+R^2)^{3/2}}$, where $dI = nIdx$

Do integral (by trig substitution) to get.. $B = \frac{1}{2}\mu_0 nI$.

Ampere's Law

By now, you recognize that the magnetic field generated by a current carrying wire forms concentric circles around the wire.



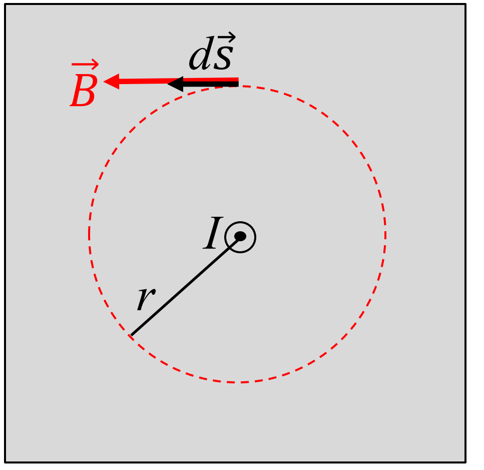


It has been found that the product of the component of $\vec{B}$ along a small step $d\vec{s}$ summed around a closed contour enclosing a current equals that enclosed current times the permeability constant. That is, $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$. This is Ampere's law.

Ampere's Law Example #1

Revisiting the long straight wire...

Calculate the magnetic field at a distance r from the wire. Note that at any point on a given field line, $\vec{B}$ and $d\vec{s}$ are parallel and that $|\vec{B}|$ is a constant. Choose an "Amperian loop" that takes advantage of this symmetry: a circle or radius r.



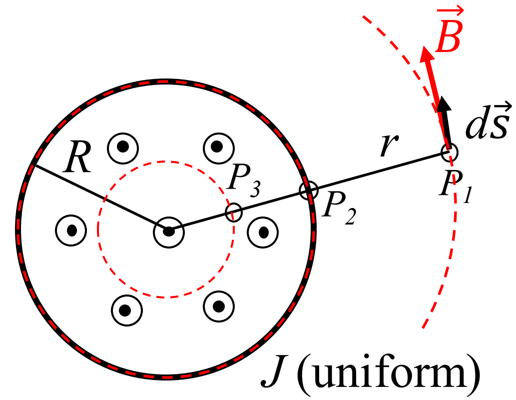
Then $\oint \vec{B} \cdot d\vec{s} = \oint B ds = B \oint ds = B(2\pi r)$

The current enclosed is just I. So, $B(2\pi r) = \mu_0 I$. Solve for $B = \frac{\mu_0 I}{2\pi r}$.

This is the same result found using the Biot-Savart law!

Ampere's Law Example #2

Consider a circular wire with a uniform current density carrying a total current I. If the wire has a radius R, calculate the $\vec{B}$-field at a distance r from the wire for $r > R$ (outside), $r = R$ (on the surface), and r < R (inside the wire).



As in the previous example, $\vec{B}$ and $d\vec{s}$ are parallel and that $|\vec{B}|$ is a constant at any point on a given field line. Choosing a circular Amperian loop,

$$\oint \vec{B} \cdot d\vec{s} = \oint B ds = B \oint ds = B(2\pi r)$$

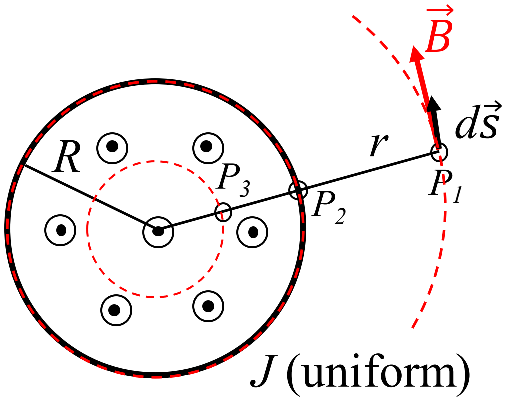
for all three points, P_1, P_2, and P_3.

Ampere's Law Example #2 cont'd

At P_1 ($r > R$): $I_{enc} = I$, so $B = \frac{\mu_0 I}{2\pi r}$.

At P_2 ($r = R$): $I_{enc} = I$, so $B = \frac{\mu_0 I}{2\pi R}$.

At P_3 ($r < R$): $I_{enc} = JA_{loop}$, where



$$J = \frac{I}{A_{wire}} = \frac{I_{enc}}{A_{loop}}.$$

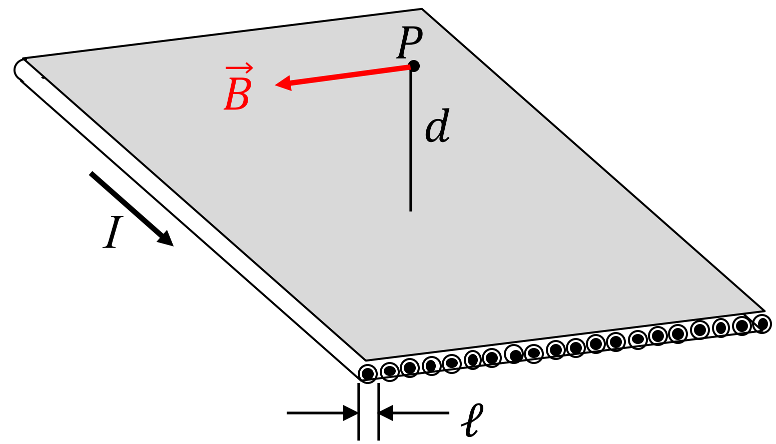
Then $I_{enc} = I\frac{\pi r^2}{\pi R^2}$ and $B(2\pi r) = \mu_0 I_{enc} = \mu_0 I \frac{r^2}{R^2}$

Solve for $B = \frac{\mu_0 I r}{2\pi R^2}$.

Example #3

Revisit the infinite sheet…

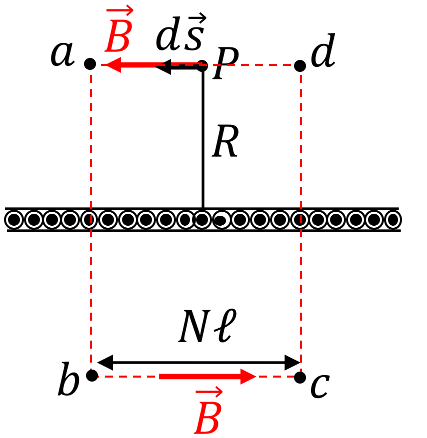
Calculate the $\vec{B}$-field a distance d from a thin infinite sheet that Carries a current I per length ℓ.



Recall that by symmetry, the $\vec{B}$-field at *any point* above the sheet to directed to the left. Using a similar argument, the $\vec{B}$-field below the sheet is directed to the the right.

Example #3 cont'd

To use Ampere's law for this situation, choose a rectangular Amperian loop that passes through point P and passes through the sheet equally on each side of the sheet. By traversing the loop counter-clockwise,



The $\vec{B}$ -field is either paralle or perpendicular to the $d\vec{s}$ steps.

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^c \vec{B} \cdot d\vec{s} + \int_c^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s} = 2B\ell$$

$0 \ (\vec{B} \perp d\vec{s})$ $\quad$ $B\ell \ (\vec{B} \parallel d\vec{s})$ $\quad$ $0 \ (\vec{B} \perp d\vec{s})$ $\quad$ $B\ell \ (\vec{B} \parallel d\vec{s})$

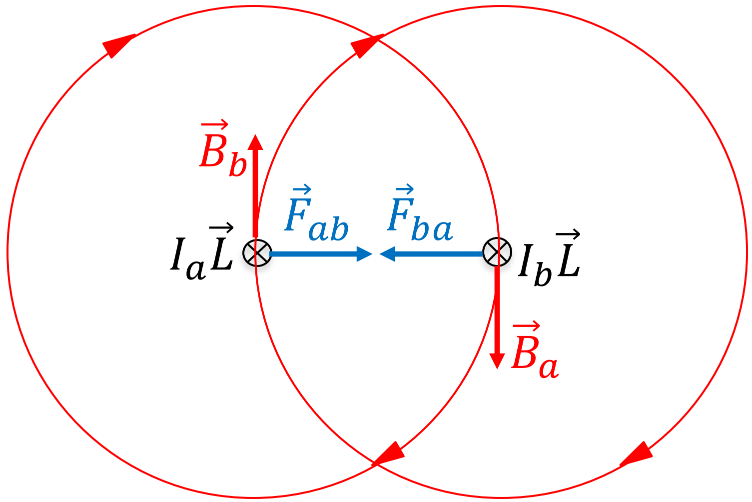
$\therefore 2B\ell = \mu_0 I_{enc} = \mu_0 n\ell I$, where n is the linear density of the "wires."

Solve for $B = \dfrac{\mu_0 n\ell I}{2\ell}$ or $B = \dfrac{\mu_0 nI}{2}$.

(Note that this is the same expression found using the Biot-Savart law!)

Forces Between Parallel Currents

Consider two parallel wires (a and b) carrying currents Ia and Ib in the same direction. Since each current generates a magnetic firld at the location of the other wire, these two wires exert forces on each other.



The magnetic force on wire *b* due to the $\vec{B}$-field generated by wire *a*:

$$\vec{F}_{ba} = I_b\vec{L} \times \vec{B}_a$$

$$\vec{F}_{ba} = I_b L B_a = I_b L \frac{\mu_0 I_a}{2\pi d} \text{ (left)}$$

The magnetic force on wire *a* due to the $\vec{B}$-field generated by wire *b*:

$$\vec{F}_{ab} = I_a\vec{L} \times \vec{B}_b$$

$$\vec{F}_{ba} = I_a L B_b = I_a L \frac{\mu_0 I_b}{2\pi d} \text{ (right)}$$

Note that $\vec{F}_{ba} = -\vec{F}_{ba}$

Punchline:
Parallel currents attract each other.

(A similar analysis would show that opposite currents repel each other.)