Electric Potential
&
Electric Potential Energy
The force on a test charge $q_0$ in an $\mathbf{E}$-field:

$$\vec{F}_{q_0,\text{field}} = q_0 \vec{E}$$

To move a charge around in an electric field, work is by the field (and by an external agent).

$$W_{\text{field}} = -W_{\text{ext. agent}}$$

Recall:

$$dW_{\text{field}} = \vec{F}_{q_0,\text{field}} \cdot d\vec{s}$$

So…

$$W_{\text{field}} = q_0 \int_A^B \vec{E} \cdot d\vec{s} = -\Delta U$$

where $d\vec{s}$ is a differential displacement along some path from Point $A$ to Point $B$.

Since $\mathbf{F}_{\text{field}}$ is a conservative force, what does this imply about how $\Delta U$ depends on the path taken from Point $A$ to Point $B$?

No path dependence!
A mechanical analogy may be helpful…

Because “opposites” attract and “likes” repel, assemblies of charge possess (electric) potential energy.

We’ll see that like *universal gravitation*, it is often convenient to define \( U = 0 \) when the charges are \( \infty \) apart.
Defining *Potential*

Again, consider a mechanical analogy: Gravitational Potential

We can define gravitational potential, “*GP***”, as the “gravitational potential energy per unit mass.”

That is,  
\[ GP = \frac{U_g}{m} = \frac{mgh}{m} = gh \]

Note that *GP* depends only on location in space. There is no gravitational potential *energy* until a mass *m* is placed at a location where the gravitational potential is nonzero.

Twice the mass, then twice the *U*<sub>g</sub> and twice the work to move it. Ten times the mass, then ten times the *U*<sub>g</sub> and ten times the work.
Defining *Electric Potential*

Similarly, moving a particle with *twice the charge* in an electric field requires *twice as much work*.

Moving a particle with *ten times the charge* means *ten times the work*.

We define *Electric Potential*, $V$, as the “electric potential energy per unit charge.” That is,

$$ V = \frac{U_E}{q_0} $$

where $V$ is measured in (Joules/Coulomb) and 1 (J/C) $\equiv$ 1 “Volt” and depends only on the location in space.
Potential Difference

\[ \Delta V = V_{\text{final}} - V_{\text{initial}} = (\Delta U_E/q_0) = (-W_{\text{field}}/q_0) \]

\[ \Delta V = -\int_A^B \vec{E} \cdot d\vec{s} \]

Again, since the \( \vec{F}_{\text{field}} \) is conservative,

\( \Delta V \) is path independent.
Example: Uniform E-field

\[ \Delta V = V_B - V_A = - \int_A^B \vec{E} \cdot d\vec{s} = -Ed \]

\[ \Delta U = q_0 \Delta V = -q_0 Ed \]

If \( q_0 > 0 \), then \( \Delta U < 0 \) (particle moves to a lower \( U \).)
Uniform \( \mathbf{E} \)-field (cont’d)

\[
\Delta V = V_C - V_A = - \int_A^C \mathbf{E} \cdot d\mathbf{s} = -E \cdot d'
\]

\[
= -Ed' \cos \theta = -Ed
\]

For path \( A \rightarrow B \rightarrow C \), \( \Delta V = -Ed + \left( - \int_B^C \mathbf{E} \cdot d\mathbf{s} \right) = -Ed
\]

\( \Delta V_{AC} = \Delta V_{AB} \) implies \( V_C = V_B \).
Equipotential Contours/Surfaces

Paths along which the electric potential is constant are referred to as **equipotential contours**.

**Question:**
If $\mathbf{E}$ points to the right, is $V$ higher or lower on the left side?

**Answer:** $V$ is higher on the left side.
Question

To move an electron from 90 V to 70 V does an external agent do positive work, negative work or zero work?

A) $W_{ext} > 0$  
B) $W_{ext} < 0$

C) $W_{ext} = 0$  
D) Not enough information.

Answer: $W_{ext} > 0$
Electric Potential from a Point Charge

\[ \Delta V = V_B - V_A = - \int_A^B \hat{E} \cdot d\hat{s}, \text{ where } \hat{E} = \frac{kq}{r^2} \hat{r}. \]

Then, \[ \Delta V = - \int_A^B \frac{kq}{r^2} \hat{r} \cdot d\hat{s} \]

where \( \hat{r} \cdot d\hat{s} = ds \cos \theta = dr \) (the projection of \( d\hat{s} \) along \( \hat{r} \)).

\[ \Delta V = - \int_A^B \frac{kq}{r^2} dr = kq \left( \frac{1}{r_B} - \frac{1}{r_A} \right). \]

Letting \( r_A = \infty \), then \( V_A = 0 \), and

\[ V_B = \Delta V_B = \frac{kq}{r_B} \]
Some things to note:

• For a point charge, \( V \) at a location \( P \) in space depends only on radial the coordinate \( r \) from the charge.

• Electric potential superposition. So, for two or more points charges:

\[
V_P = V_{1P} + V_{2P} + V_{3P} + \ldots + V_{NP} = \sum_{i=1}^{N} V_{iP} = k \sum_{i=1}^{N} \frac{q_i}{r_i}
\]

• \( V_P \) is an algebraic sum rather than a vector sum. Thus, it is generally easier to calculate \( V \) than to calculate \( \mathbf{E} \).
Electric Potential Energy

Now bring a $q_2$ from $\infty$ and place it a distance $r_{12}$ from $q_1$:

$$U_E = q_2 V_1 = \frac{kq_1 q_2}{r_{12}}$$ for the 2-particle system.

If instead you bring $q_1$ from $\infty$ and place it a distance $r_{12}$ from $q_2$, then...

$$U_E = q_1 V_2 = \frac{kq_1 q_2}{r_{12}} \text{ (the same.)}$$
Now, what happens when you then bring in a third point charge and place it at $P$ near the first two?

Then $\Delta U = U_f - U_i = q_3 V_P = q_3(V_1 + V_2)$

The total electrical potential energy of the 3-particle system is $U_f = U_i + \Delta U = U_i + q_3 V_1 + q_3 V_2$ or

$$U_f = \frac{kq_1q_2}{r_{12}} + \frac{kq_1q_3}{r_{13}} + \frac{kq_2q_3}{r_{23}}$$

To calculate the total $U_E$, you must sum over every pair of charges. $U_E$ does not obey superposition!
Electric Potential from a Dipole

At $P$: $V_P = \sum_{i=1}^{2} V_i = V_+ + V_- = k \left( \frac{q}{r_+} + \frac{-q}{r_-} \right) = kq \left( \frac{r_- + r_+}{r_+ r_-} \right)$

For $r \gg d$:
$r_- + r_+ \approx d \cos \theta$
and $r_+ r_- \approx r^2$.

Thus, $V = kq \frac{d \cos \theta}{r^2} = k \frac{p \cos \theta}{r^2}$,

since $p = qd$.

Where $r_+ = r_-$ (as in the case of any point in the $xz$-plane), $V = 0.$