Electric Flux & Gauss’ Law
Calculating Electric Fields

Previously, we learned to calculate $E$-fields of a given charge distribution using Coulomb’s Law:

$$\vec{E} = k \int \frac{dq}{r^2} \hat{r}$$

We now introduce an alternative method that is MUCH easier to use when calculating $E$-fields of highly symmetric charge distributions:

**Gauss’ Law**
But first: the concept of FLUX

Consider a stream of water (say in a river or a pipe) moving uniformly with velocity $v$, through a rectangular loop of area $A$ and perpendicular to that area.

\[ \text{Diagram showing water flow through a rectangular loop with velocity } v \text{ and area } A. \]
Let $\Phi$ represent the “volume flow rate” through the loop. That is,

$$\Phi = \frac{\text{volume}}{\text{unit time}}$$

Note that $\Phi$ depends on $v$, $A$ and the angle $\theta$ that the loop makes with the direction of flow.

If the area $A$ is not perpendicular to the flow the number of “velocity lines” passing through the loop is less.
However, the # of lines passing through A is equal to the # that passes through the projected rectangular area $A' = A \cos \theta$ which IS perpendicular to the flow.

**Note:** $\theta$ is the angle between $\mathbf{v}$ and the dashed line which is perpendicular to the plane of the loop.

Since the volume flow rate depends on the component of $\mathbf{v}$ that is normal ($\perp$) to the plane of the loop ($\mathbf{v} \cos \theta$) we can write the flow rate or **volume flux** (in this example) as

$$\Phi = \nu A' = \nu A \cos \theta = \mathbf{\nu} \cdot \mathbf{A}$$
Now consider this:

What if the velocity varied (in magnitude and direction) from point to point?

and/or

What is the area was not flat? (Imagine a crinkled wire mesh so that each little opening of the mesh had a different orientation relative to the flow.)

Then, $\Phi = \int \vec{v} \cdot d\vec{A}$
Electric Flux

Just as we can define a volume flux of a velocity field of a fluid through an area, we define electric flux (flux of an electric field) as

\[ \Phi = EA \] for a uniform field passing \( \perp \) through a (flat) area \( A \).

If \( A \) makes an angle \( \theta \) with the (uniform) field, then

\[ \Phi = EA \cos \theta \]
In general, the electric field may vary (in both magnitude and direction) across an arbitrary asymmetric surface:

In this case, the electric flux through the surface by summing over all the $\Delta A$’s where each $\Delta A$ is small enough to consider the $E$-field through that $\Delta A$ to be constant. Then,

$$
\Phi = \lim_{\Delta A_i \to 0} \sum \vec{E}_i \cdot \Delta \vec{A}_i = \int_{surf} \vec{E} \cdot d\vec{A}
$$
Flux Through a Closed Surface

\[ \Delta \Phi_i = \vec{E}_i \cdot \Delta A_i \hat{n}_i \]

By convention, choose the direction of the various surface elements to be the outward normal, \( \hat{n}_i \).

Then \( \Delta \Phi_1, \Delta \Phi \) are \( > 0 \), and \( \Delta \Phi_3 < 0 \).
Summarizing…

- Field lines that *leave* the (closed) surface produce a *positive* flux.
- Field lines that *enter* the surface produce a *negative* flux.

The net flux is $\propto (# \text{ lines leaving} - # \text{ lines entering})$ the closed surface.

So the net electric flux through a closed surface is:

$$\Phi_{\text{net}} = \oint \vec{E} \cdot d\vec{A}$$
Example

A uniform electric field points in the $+x$-direction. A cube is situated with a corner at the origin and three edges along the $+x$-, $+y$-, and $+z$-axes. Calculate the electric flux through each face of the cube and the net electric flux through the cube.
Gauss’ Law

• A much simpler way to calculate electric fields for highly symmetric charge distributions.

• Relies on the inverse-square relationship of the electric field of point charges.

• Relates the net electric flux through a closed surface (which is called a Gaussian surface) to the net electric charge enclosed by that surface.
Gauss’ Law

\[ \Phi_E = \oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0} \]

…where \( A \) is any surface that encloses a net amount of charge, \( q_{enc} \).

Gauss’ law states that \( \Phi_E \) is independent of the location of the charge within the surface.

This result applies to included several charges or even continuous distributions of charge within the surface. (That is, \( q_{enc} = \sum_i q_i \) within the surface.)
Continuous Distribution in Surface?

If there is a charge density $\rho$ within the volume $V$ enclosed by the surface, then the total charge enclosed is:

$$q_{enc} = \int \rho dV$$

So, now Gauss’ law is now in general form is:

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int \rho dV$$
Examples...

• Charge uniformly spread on the surface of a sphere ($\sigma =$ constant).
• Charge uniformly spread throughout the volume of a sphere ($\rho =$ constant).
• An infinitely long wire with a uniform (linear) charge density, $\lambda$.
• An infinite planar sheet with a uniform (surface) charge density, $\sigma$. 
Conductors in Electrostatic Equilibrium

- Due to electrical repulsion, ALL excess charge must reside on the surface of the conductor.
- No currents inside (electrostatic equilibrium).
- \( \Phi_E = 0 \), so \( q_{enc} = 0 \). Thus, \( E = 0 \) inside.
- \( E \)-field just outside the surface of a conductor is \( \sigma/\varepsilon_0 \), not \( \sigma/2\varepsilon_0 \). (Why?)