Electric Flux
&
Gauss' Law

Calculating Electric Fields

Previously, we learned to calculate **E**-fields of a given charge distribution using Coulomb's Law:

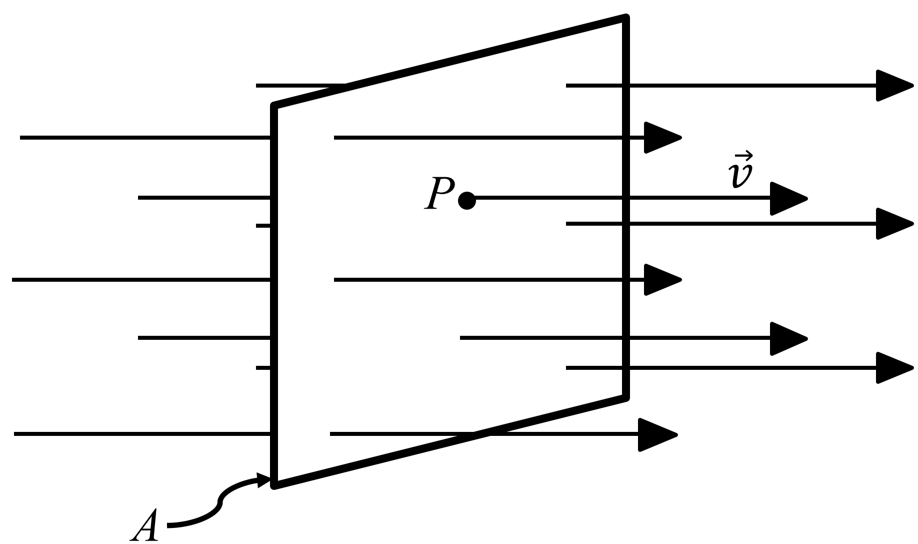
$$\vec{E} = k \int \frac{dq}{r^2} \hat{r}$$

We now introduce an alternative method that is MUCH easier to use when calculating **E**-fields of highly symmetric charge distributions:

Gauss' Law

But first: the concept of FLUX

Consider a stream of water (say in a river or a pipe) moving uniformly with velocity **v**, through a rectangular loop of area *A* and perpendicular to that area.



Let Φ represent the "volume flow rate" through the loop. That is,

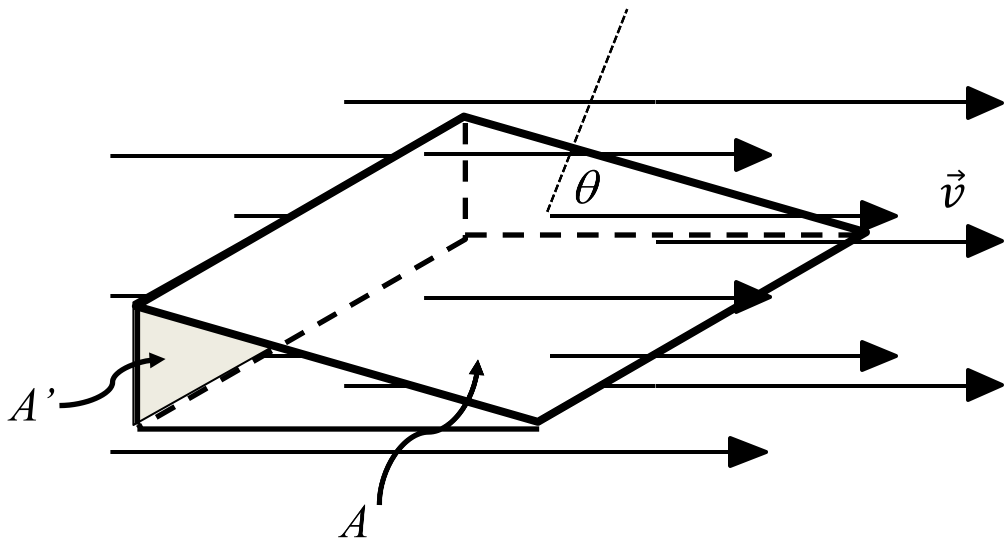
$$\Phi = \frac{\text{volume}}{\text{unit time}}$$

Note that Φ depends on v, A and the angle θ that the loop makes with the direction of flow.

If the area A is not perpendicular to the flow the number of "velocity lines" passing through the loop is less.

However, the # of lines passing through A is equal to the # that passes through the projected rectangular area $A' = A\cos\theta$ which IS perpendicular to the flow.

Note: θ is the angle between **v** and the dashed line which is perpendicular to the plane of the loop.



Since the volume flow rate depends on the component of **v** that is normal (⊥) to the plane of the loop ($v\cos\theta$) we can write the flow rate or volume flux (in this example) as

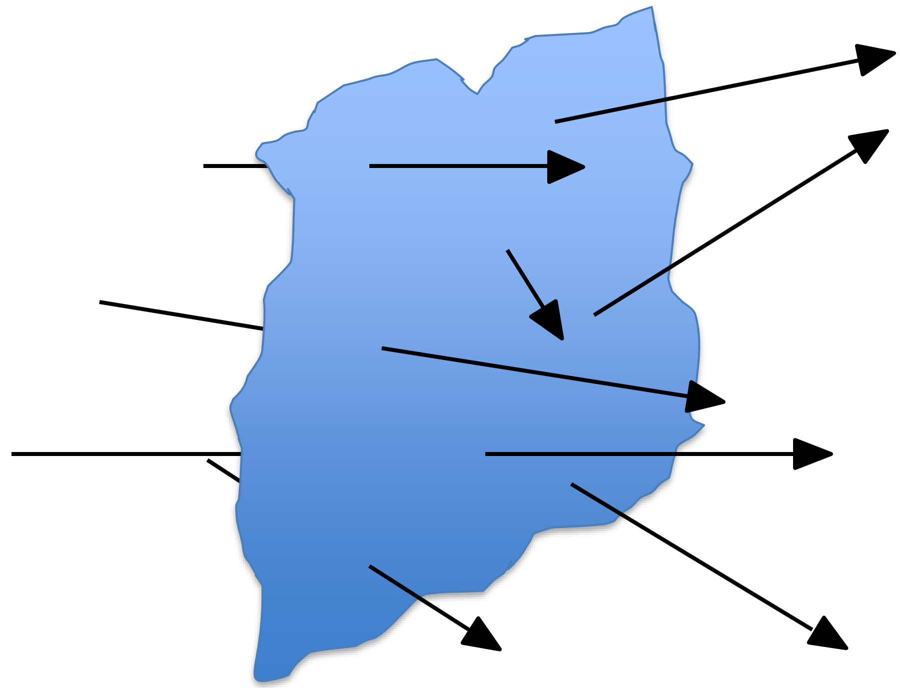
$$\Phi = vA' = vA\cos\theta = \vec{v}\cdot\vec{A}$$

Now consider this:

What if the velocity varied (in magnitude and direction) from point to point?

and/or

What is the area was not flat? (Imagine a crinkled wire mesh so that each little opening of the mesh had a different orientation relative to the flow.)



Then, $\Phi = \int \vec{v} \cdot d\vec{A}$

Electric Flux

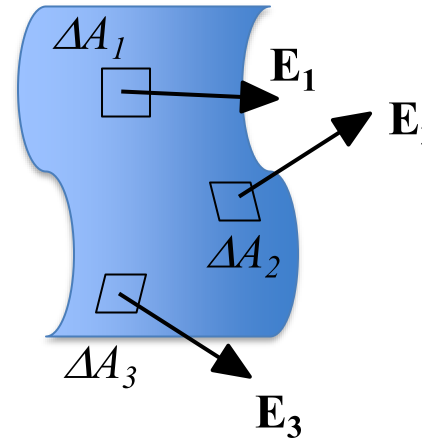
Just as we can define a volume flux of a velocity field of a fluid through an area, we define electric flux (flux of an electric field) as

$\Phi = EA$ for a uniform field passing $\perp$ through a (flat) area A.

If A makes an angle θ with the (uniform) field, then

$$\Phi = EA\cos\theta$$

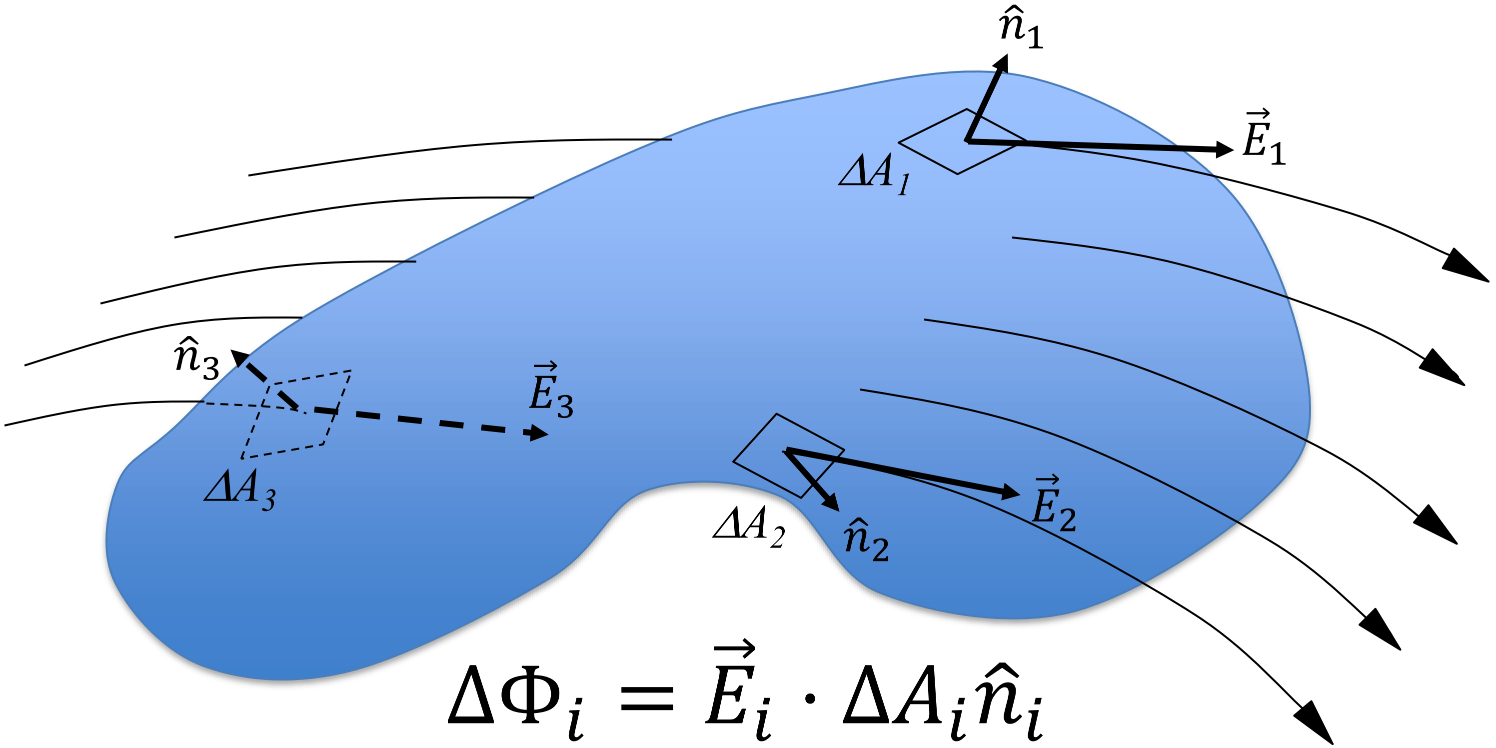
In general, the electric field may vary (in both magnitude and direction) across an arbitrary asymmetric surface:



In this case, the electric flux through the surface by summing over all the *ΔA*'s where each *ΔA* is small enough to consider the **E**-field through that *ΔA* to be constant. Then,

$$\Phi = \lim_{\Delta A_i \to 0} \sum \vec{E}_i \cdot \Delta\vec{A}_i = \int_{surf} \vec{E} \cdot d\vec{A}$$

Flux Through a Closed Surface



$$\Delta\Phi_i = \vec{E}_i \cdot \Delta A_i \hat{n}_i$$

By convention, choose the direction of the various surface elements to be the outward normal, $\hat{n}_i$.

Then $\Delta\Phi_1$, $\Delta\Phi$ are > 0 , and $\Delta\Phi_3 < 0$.

Summarizing…

- Field lines that *leave* the (closed) surface produce a *positive* flux.
- Field lines that *enter* the surface produce a *negative* flux.

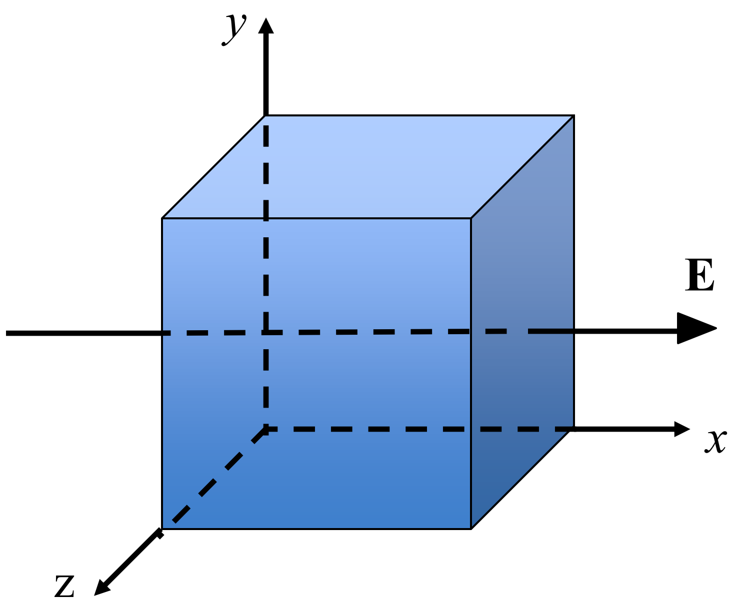
The net flux is $\propto$ (# lines leaving – # lines entering) the closed surface.

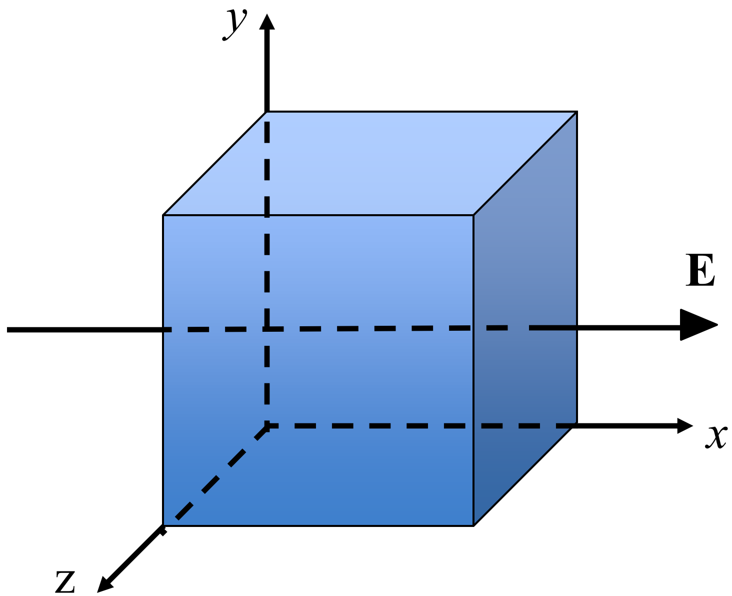
So, the net electric flux through a closed surface is:

$$\Phi_{net} = \oint \vec{E} \cdot d\vec{A}$$

Example

A uniform electric field points in the +*x*-direction. A cube is situated with a corner at the origin and three edges along the +*x*-, +*y*-, and +*z*-axes. Calculate the electric flux through each face of the cube and the net electric flux through the cube.





$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

$$\Phi_E = \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{front}} + \int_{\text{back}}$$

$$\int_{\text{left}} = EA\cos(180°) = -EA$$

$$\int_{\text{right}} = EA\cos(0°) = EA$$

$$\int_{\text{top}} = \int_{\text{top}} = \int_{\text{top}} = \int_{\text{top}} = EA\cos(90°) = 0$$

So, $\Phi_E = 0$ through the cube.

Gauss’ Law

- A much simpler way to calculate electric fields for highly symmetric charge distributions.
- Relies on the inverse-square relationship of the electric field of point charges.
- Relates the net electric flux through a closed surface (which is called a Gaussian surface) to the net electric charge enclosed by that surface.

Gauss' Law

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$$

…where A is <u>any</u> surface that encloses a net amount of charge, q_{enc}.

Gauss' law states that Φ_E is independent of the location of the charge within the surface.

This result applies to included several charges or even continuous distributions of charge within the surface. (That is, $q_{enc} = \Sigma_i q_i$ within the surface.)

Continuous Distribution in Surface?

If there is a charge density ρ within the volume V enclosed by the surface, then the total charge enclosed is:

$$q_{enc} = \int \rho dV$$

So, now Gauss' law is now in general form is:

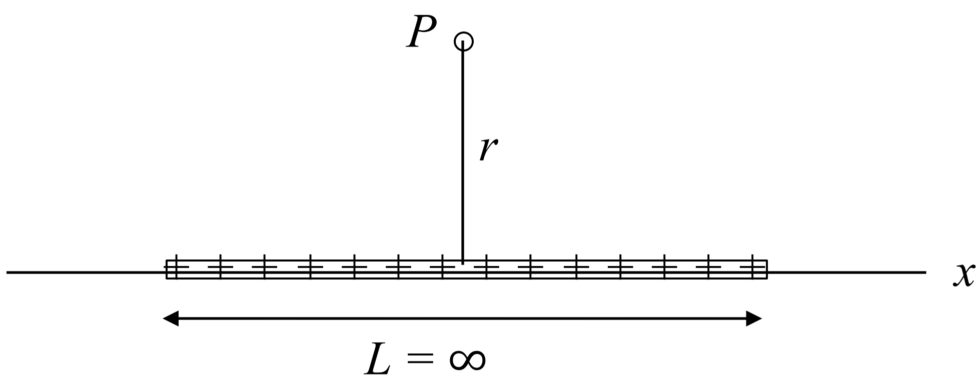
$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int \rho dV$$

Examples to consider…

1. An infinitely long wire with a uniform (linear) charge density, λ.
2. An infinite planar sheet with a uniform (surface) charge density, σ.
3. Charge uniformly spread throughout the volume of a sphere (ρ = constant).
4. Charge uniformly spread on the surface of a sphere (σ = constant).

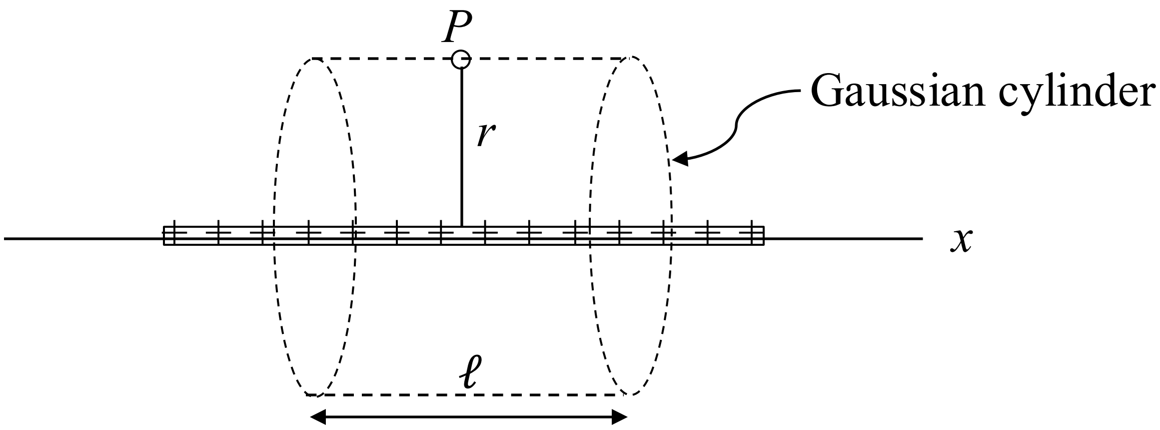
Example #1

Infinite line of uniform charge density λ.



Calculate the **E**-field for a point located a distance r from the line.

- The infinite line of uniform charge has cylindrical symmetry.
- Construct a "Gaussian surface" of arbitrary length h with Point P on the surface of the cylinder.



$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \int_{\text{left cap}} + \int_{\text{right cap}} + \int_{\text{cyl body}}$$

$$\int_{\text{left cap}} = \int_{\text{right cap}} = 0, \text{ since } \vec{E} \perp d\vec{A}$$

$$\text{and } \int_{\text{cyl body}} = \int E dA = E \int dA = EA = E(2\pi r \ell).$$

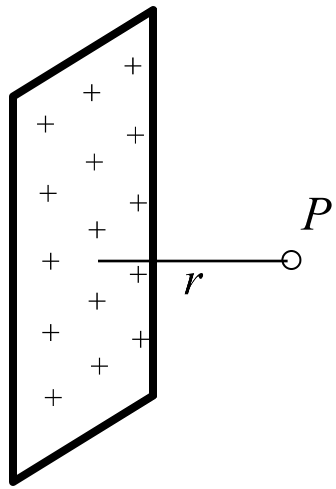
$$\text{Gauss' Law concludes: } E(2\pi r \ell) = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}.$$

$$\text{So, } E(2\pi r \ell) = \frac{\lambda \ell}{\varepsilon_0}.$$

$$\vec{E} = \frac{\lambda}{2\pi r \varepsilon_0} \hat{r}$$

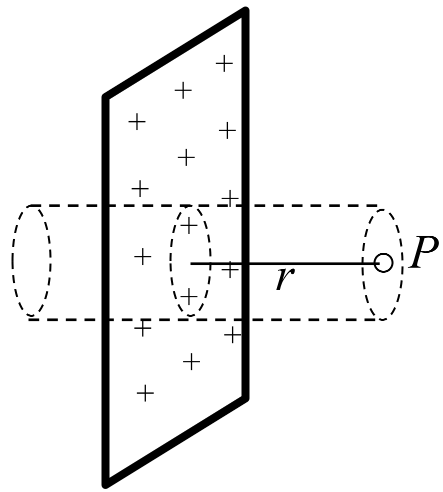
Example #2

Uniformly charged insulating sheet having uniform (surface) charge density σ.



Calculate the **E**-field for a point a distance r from the sheet.

- The infinite line of uniform charge has planar symmetry.
- Construct a "Gaussian" cylinder (or box or prism) of arbitrary end cap area A extending perpendicularly through the sheet equally with Point P on the surface of the cylinder.



$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \int_{\text{left cap}} + \int_{\text{right cap}} + \int_{\text{cyl body}}$$

$$\int_{\text{left cap}} = \int_{\text{right cap}} = EA, \text{ since } \vec{E} \parallel d\vec{A}$$

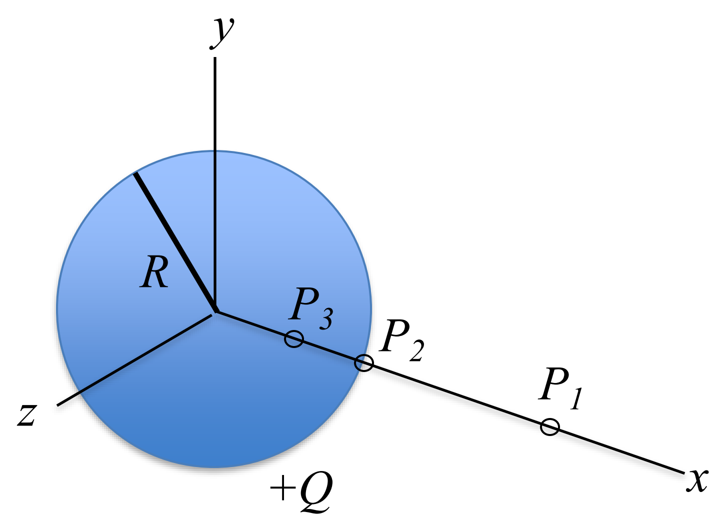
$$\text{and } \int_{\text{cyl body}} = 0, \text{ since } \vec{E} \perp d\vec{A}.$$

$$\text{Gauss' Law concludes: } 2EA = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}.$$

So, $\vec{E} = \frac{\sigma}{2\varepsilon_0}$, directed perpendicularly away from the sheet.

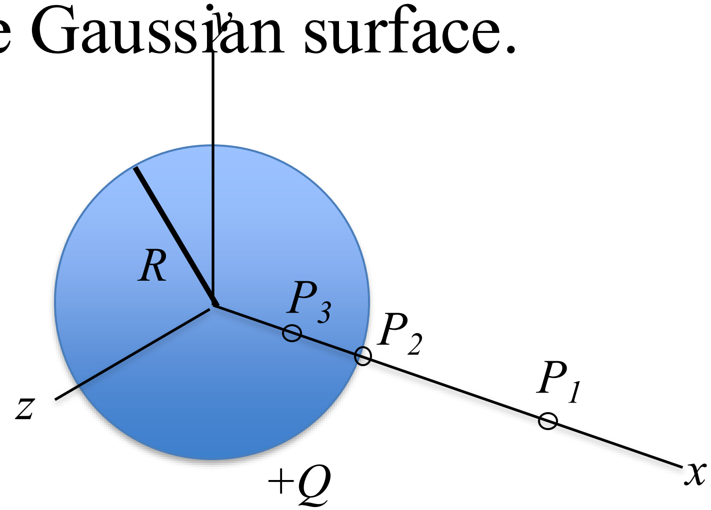
Example #3

Uniformly charged insulating sphere (having total charge $+Q$ and radius R).



Calculate the **E**-field for a point outside, on the surface, and inside the sphere.

- The sphere of uniform charge density has spherical symmetry.
- Construct a "Gaussian sphere" of radius r centered on the center of the insulating sphere with the points in question on the Gaussian surface.



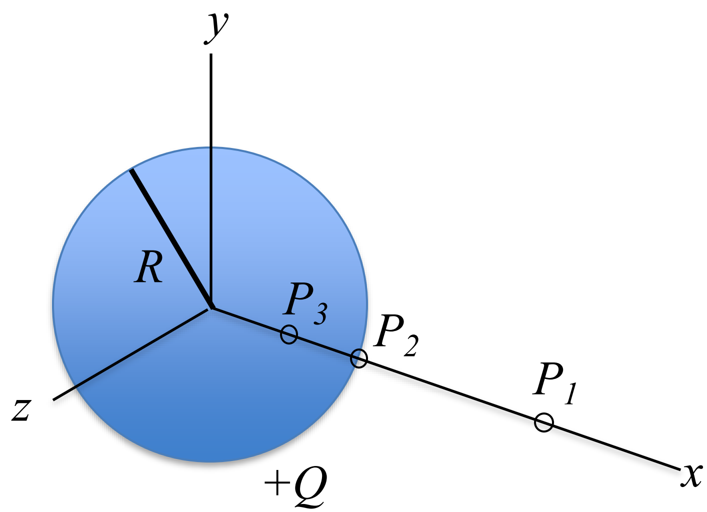
Note that $\vec{E}$ and $d\vec{A}$ are parallel at all points on the chosen Gaussian surface regardless of the value of r.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E dA = E \oint dA = E A_{sphere} = E(4\pi r^2).$$

Gauss' Law concludes: $E(4\pi r^2) = \frac{q_{enc}}{\varepsilon_0}$ for all three points.

$$\vec{E} = \frac{q_{enc}}{4\pi r^2 \varepsilon_0} \hat{r}.$$

However, q_{enc} does depend on the value r…



For $r_1 > R$ and $r_2 = R$, $q_{enc} = +Q$.

$$\vec{E}_{P_1} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r} \text{ for } r > R.$$

$$\vec{E}_{P_2} = \frac{Q}{4\pi R^2 \varepsilon_0}\hat{r} \text{ for } r = R.$$

For $r_3 < R$: q_{enc} = (charge density)X(volume of Gaussian sphere).

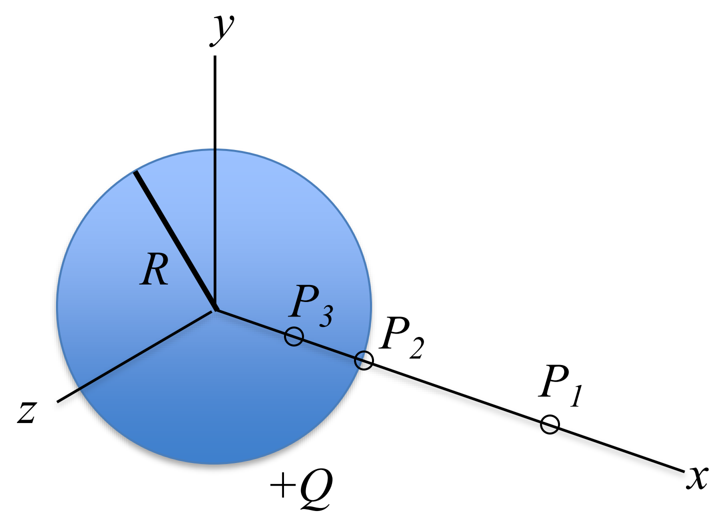
$$q_{enc} = \rho\frac{4}{3}\pi r^3, \text{ where } \rho = \frac{Q}{\frac{4}{3}\pi R^3}.$$

Gauss' Law concludes: $E(4\pi r^2) = \dfrac{\rho\frac{4}{3}\pi r^3}{\varepsilon_0}$.

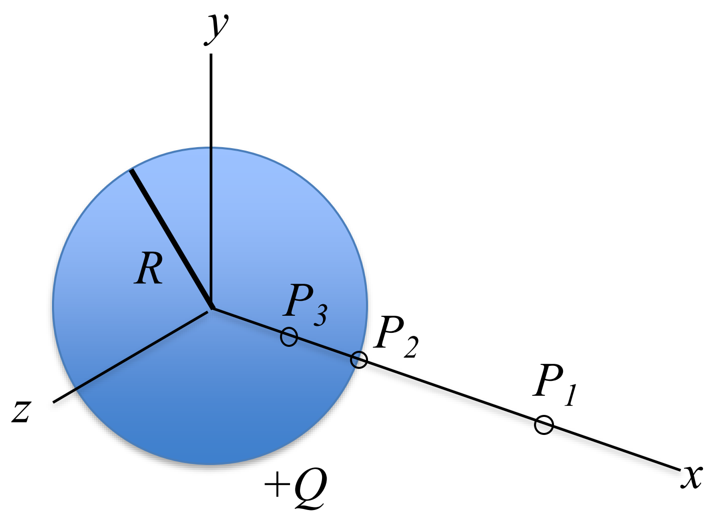
$\vec{E} = \dfrac{\rho r}{3\varepsilon_0}\hat{r}$ or $\vec{E} = \dfrac{kQr}{R^3}\hat{r}$ inside the insulating sphere.

Example #4

Uniformly charged conducting sphere (having total charge $+Q$ and radius R).



Calculate the **E**-field for a point outside, on the surface, and inside the sphere.



For $r_1 > R$ and $r_2 = R$, $q_{enc} = +Q$.

(No difference from Example 3.)

$$\vec{E}_{P_1} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r} \text{ for } r > R.$$

$$\vec{E}_{P_2} = \frac{Q}{4\pi R^2 \varepsilon_0}\hat{r} \text{ for } r = R.$$

For $r_3 < R$: $q_{enc} = 0$, since for a conducting sphere all the excess charge resides on the outer surface.

Gauss' Law concludes: $E(4\pi r^2) = \frac{0}{\varepsilon_0}$.

$\vec{E} = 0$ inside the sphere.

Conductors in Electrostatic Equilibrium

- Due to electrical repulsion, ALL excess charge must reside on the surface of the conductor.
- No currents inside (electrostatic equilibrium).
- $\Phi_E = 0$, so $q_{enc} = 0$. Thus, **E** = 0 inside.
- **E**-field just outside the surface of a conductor is σ/ε_0, not $\sigma/2\varepsilon_0$. (Why?)