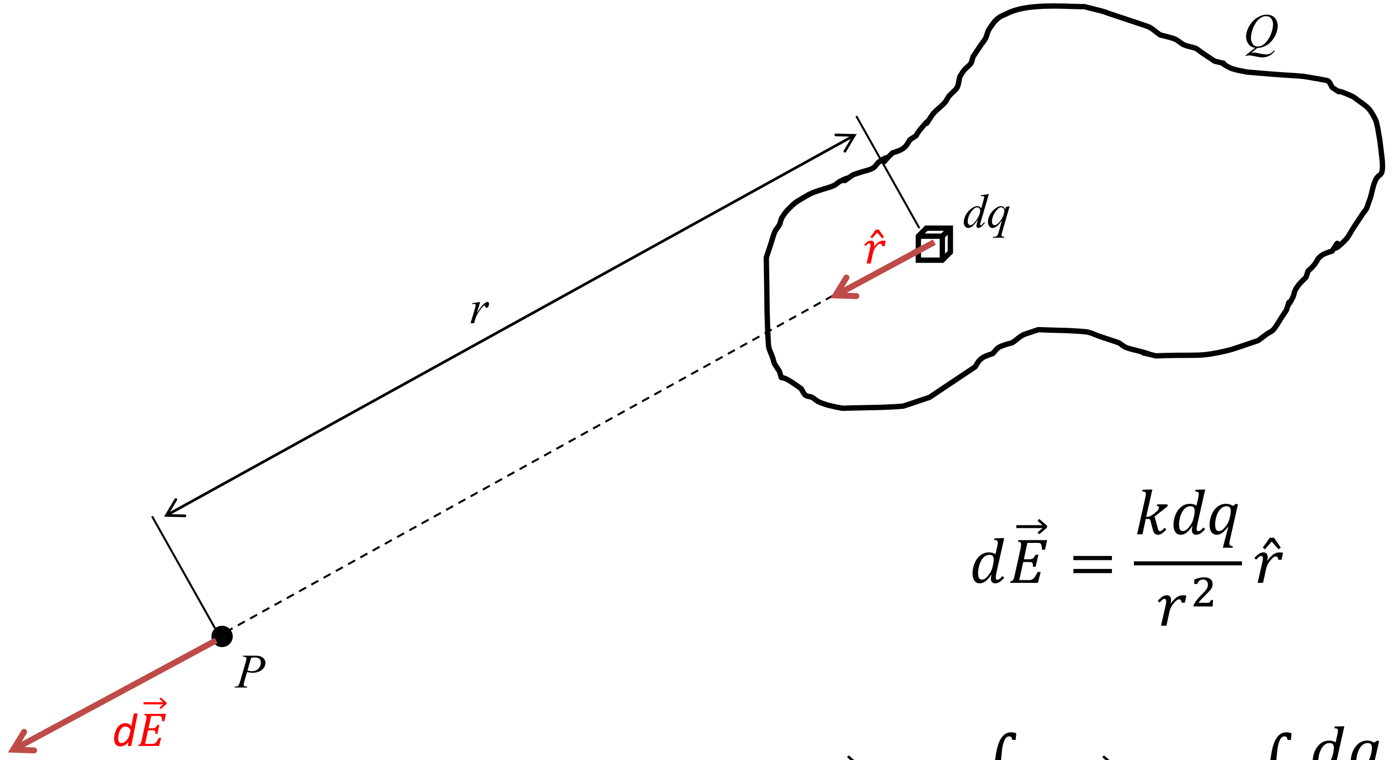
Continuous Distributions of Charge



$$d\vec{E} = \frac{k\,dq}{r^2}\hat{r}$$

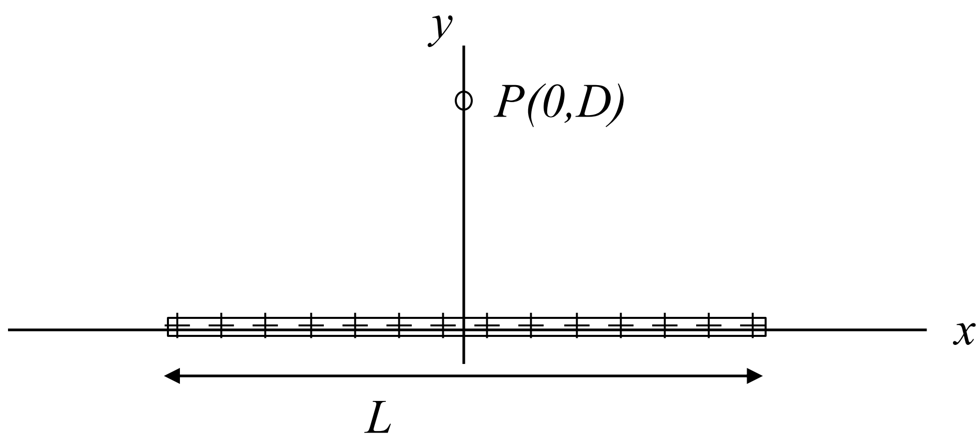
$$\vec{E}_P = \int d\vec{E} = k\int \frac{dq}{r^2}\hat{r}$$

Charge Density

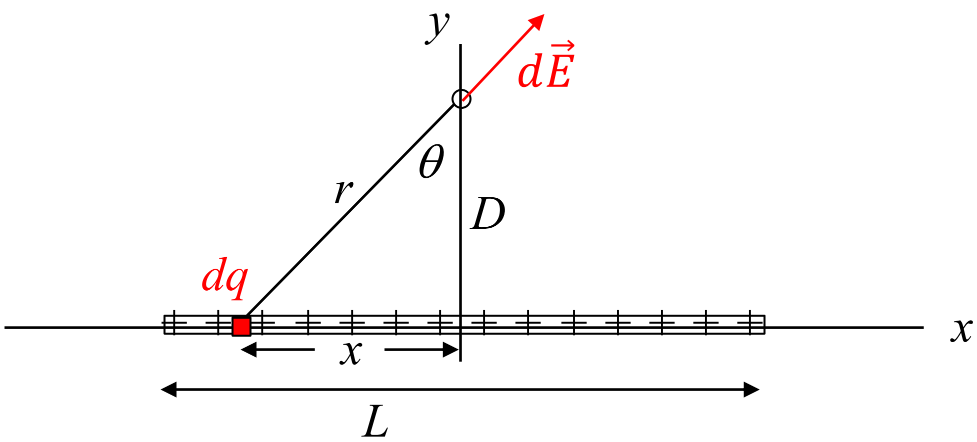
Type	**Uniform**	**Nonuniform**
Linear	$\lambda = \frac{Q}{L}$	$\lambda = \frac{dq}{d\ell}$
Surface	$\sigma = \frac{Q}{A}$	$\sigma = \frac{dq}{dA}$
Volume	$\rho = \frac{Q}{V}$	$\rho = \frac{dq}{dV}$

Example #1

Uniform Finite Line of Charge (having total charge Q and length L (centered about the y-axis).



Calculate the **E**-field for a point on the y-axis located a distance D from the line.



$$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath}$$

By symmetry: $E_x = 0$

Note: $r = (x^2 + D^2)^{½}$ and $\cos\theta = \frac{D}{(x^2+D^2)}$ and $dq = \lambda dx$

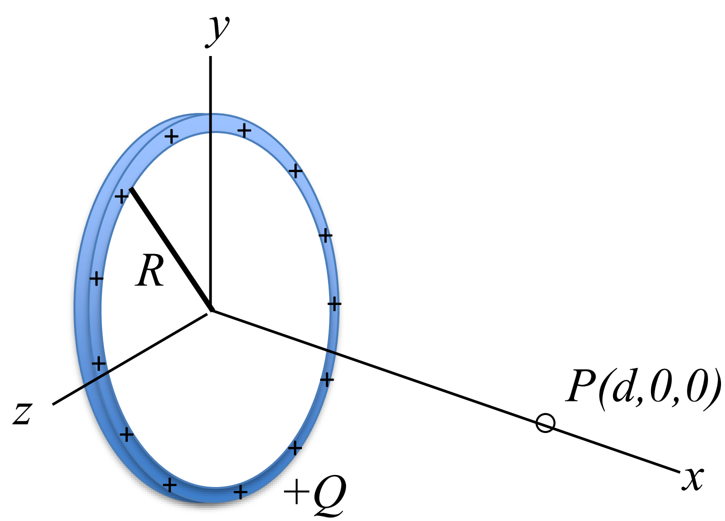
$$dE_y = dE\cos\theta = \frac{kdq}{(x^2+D^2)}\frac{D}{(x^2+D^2)}.$$

Integrate from –L/2 to +L/2: $E_y = \int_{-L/2}^{+L/2} dE_y = \frac{2kQ}{LD}\left(1 + \frac{4D^2}{L^2}\right)^{-1/2}$

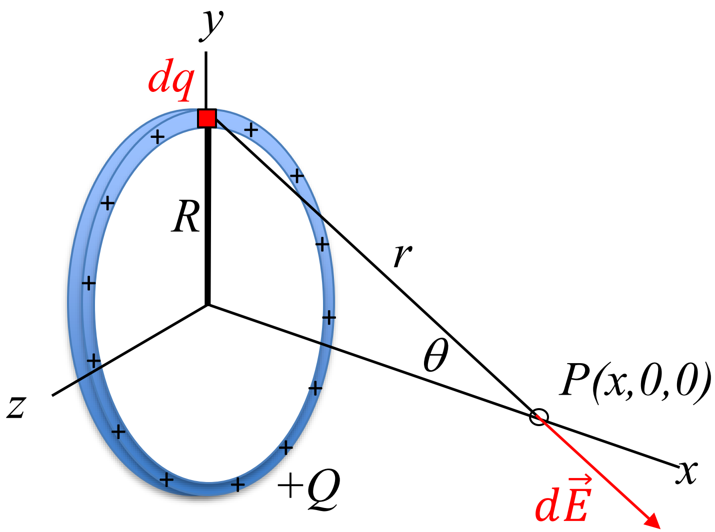
$$\vec{E} = \frac{2kQ}{LD}\left(1 + \frac{4D^2}{L^2}\right)^{-1/2}\hat{\jmath}$$

Example #2

Uniform Ring of Charge (having total charge $+Q$ and radius R).



Calculate the **E**-field for a point on the symmetry axis (taken to be the x-axis).



$$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + +E_z\hat{k}$$

By symmetry: $E_y = E_z = 0$

$$dE = \frac{kdq}{r^2} \quad \text{and} \quad \cos\theta = \frac{x}{r}$$

$$r = (x^2 + R^2)^{1/2}$$

$$E_x = \int dE_x = \int dE\cos\theta = \int \frac{kdq}{r^2}\frac{x}{r}.$$

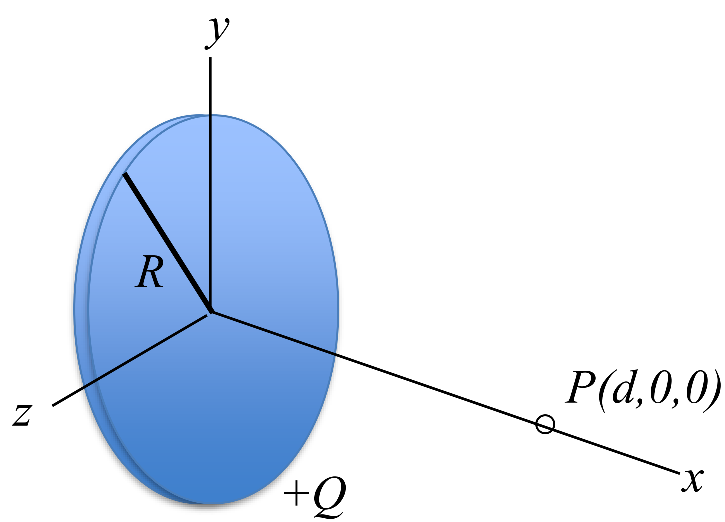
Note that r and θ are constant. So, $dE_x = \frac{kx}{r^3}\int_0^Q dq$.

Integrate over Q: $E_x = \frac{kx}{(x^2+R^2)^{3/2}}\int_0^Q dq = = \frac{kQx}{(x^2+R^2)^{3/2}}$

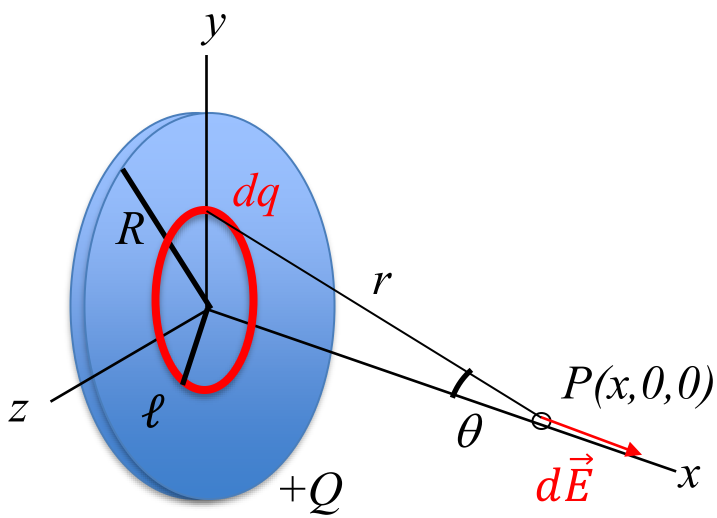
$$\vec{E} = \frac{kQx}{(x^2+R^2)^{3/2}}\,\hat{\imath}$$

Example #3

Uniform Circular Disk of Charge (having total charge $+Q$ and radius R).



Calculate the **E**-field for a point on the symmetry axis (taken to be the *x*-axis).



$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + +E_z\hat{k}$

By symmetry: $E_y = E_z = 0$

Use the previous result (ring) as the dq:

$dE_x = \frac{kx\,dq}{(x^2+\ell^2)^{3/2}}$, where $dq = (\sigma)(2\pi\ell\, d\ell)$

and the (areal) charge density $\sigma = \frac{Q}{\pi R^2}$.

Note that $r = (x^2 + \ell^2)^{½}$.

$$\text{So, } E = \int dE_x = \int \frac{2\pi k\sigma x\, \ell d\ell}{(x^2+\ell^2)^{3/2}}.$$

Integrate over ℓ from 0 to R: $E = 2kx\pi\sigma \int_0^R \frac{\ell d\ell}{(x^2+\ell^2)^{3/2}} = 2kx\pi\sigma \left(\frac{-1}{(x^2+R^2)^{½}}\right)\Big|_0^R$

$$\vec{E} = \frac{2kQ}{R^2}\left(1 - \frac{x}{(x^2+R^2)^{½}}\right)\hat{\imath} \quad \text{or} \quad \vec{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{x}{(x^2+R^2)^{½}}\right)\hat{\imath}$$

Strategy for Calculating the Electric Field due to a Continuous Distribution of Charge

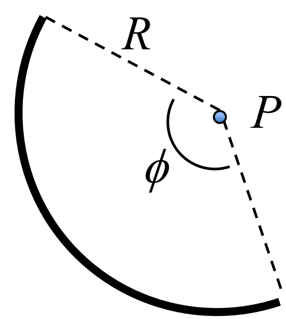
1. Divide the *Q* into *dq* elements.
2. Look for and exploit symmetry.
3. Apply superposition:

$$\vec{E}_{net} = \sum_i \vec{E}_i \rightarrow \int d\vec{E} = k \int \frac{dq}{r^2}\,\hat{r}$$

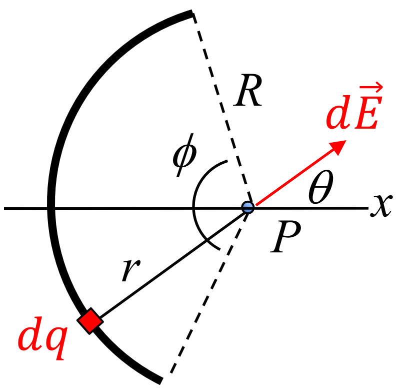
   Replace *dq* with an equivalent expression involving charge density and a geometric quantity (e.g., λdx, σdA, ρdV). All angles and distribution elements must be in terms of the integration variable.
4. Evaluate the integral and check validity in extreme limits.

Example #4

Uniform Circular Arc of Charge (of total angle ϕ and having total charge $+Q$ and radius R).



Calculate the **E**-field at the center of the circle generated by the arc.



To exploit symmetry, choose an axis that bisects the arc.

Use a (linear) charge density, λ:

$$\lambda = \frac{Q}{s} = \frac{Q}{R\phi}$$

$$\vec{E} \;=\; E_x\hat{\imath} + E_y\hat{\jmath}$$

By symmetry: $E_y = 0$

Note: $r = R =$ constant and $dq = \lambda ds = \lambda R d\theta.$

$$E_x = \int dE_x = \int dE\cos\theta = \frac{k\lambda R}{R^2}\int \cos\theta\, d\theta.$$

Integrate from $-\phi/2$ to $+\phi/2$: $E_x = \frac{k\lambda}{R}\sin\theta\Big|_{-\phi/2}^{+\phi/2}$

$\vec{E} = \frac{2k\lambda}{R}\sin\frac{\phi}{2}\,\hat{\imath}$ or $\vec{E} = \frac{2kQ}{R^2\phi}\sin\frac{\phi}{2}\,\hat{\imath}$