DIY: *Functions: Inverses*

To review function inverses, watch the following set of YouTube videos explaining what inverses are, which functions have inverses, and how to find them. The videos are followed by several practice problems for you to try, covering all the basic concepts covered in the videos, with answers and detailed solutions. Some additional resources are included for more practice at the end.

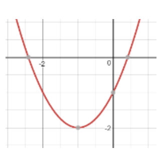
1. https://www.youtube.com/watch?v=nSmFzOpxhbY Basic of Function Inverses
   - (note: at 9:21 minutes on this video, the presenter states that because the function f(x) does not pass the horizontal line test, "it" is not a function. What he intended was to state that the inverse is not a function.)
   - If a function passes not only the vertical line test but also the horizontal line test, it is said to be a "one-to-one function".
2. https://www.youtube.com/watch?v=Ec5YYVxyq44 Finding the inverse of a function, ex. 1
3. https://www.youtube.com/watch?v=Bq7Jd7vXC08 Finding the inverse of a function, ex. 2

   (an example of a function which does not have an inverse function).
4. https://www.youtube.com/watch?v=BmjbDINGZGg Finding an inverse of a rational function
5. https://www.youtube.com/watch?v=cORGveoJoFg Finding the inverse of a radical function
6. https://www.youtube.com/watch?v=S4AEZElTPDo Composition of functions
7. https://www.youtube.com/watch?v=4uu-zIx9cCM Another introduction to inverse functions, using the concept of composite functions

Practice problems: The following problems use the techniques demonstrated in the above videos. The answers are given after the problems. Then detailed solutions, if you need them, are provided after the answer section. For further assistance and help please contact ***Math Assistance Area***.

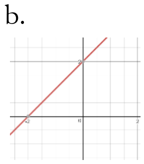
1. Determine which of the following functions are one-to-one functions (pass the horizontal line test) and so have inverse functions.

a.



y=$(x+1)^2-2$

b.

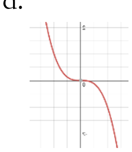


$y=x+2$

c.

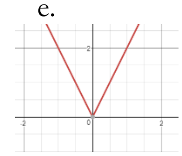
$y=x^3+2x^2-x-1$

d.

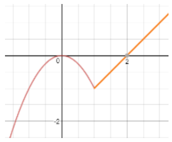


$y=-x^3$

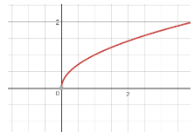
e.



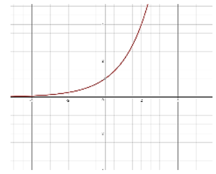
$y=|2x|$

f. 

(as shown)

g. 

$y = \sqrt{x}$

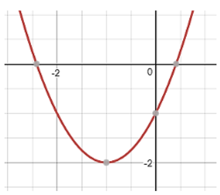
h. 

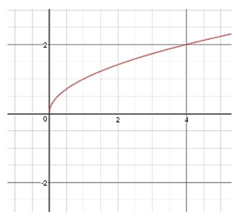
$y = 2^x$

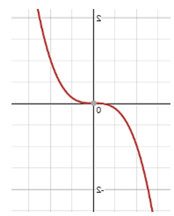
2. For each of the functions in ex. 1, what are the domain and range? For those which are one-to-ones functions, what would be the domain and range of each inverse?

3. For each of the functions in ex. 1 for which the inverse function exists, find the inverse.

4. For each of the functions graphed below, sketch the inverse function or state that inverse is not a function (the inverse function does not exist).

a. 

b. 

c. 

5. Given: $f(x) = 3x - 2$, find the inverse $f^{-1}(x)$. Then prove these functions are inverses using composition of functions.

6. Use composition of functions to determine if the following pairs of functions are inverses of each other.

a. $f(x) = \frac{3}{4}x + 8 \;\; and \;\; g(x) = \frac{-4}{3}x - 8$

b. $f(x) = \sqrt[3]{x-6} + 3 \;\; and \;\; g(x) = (x-3)^3 + 6$

Answers:

1. b, d, g, and h are one-to-one

2. a. D: (-∞,∞), R: [-2, ∞); (no inverse function)

b. D: (-∞,∞), R: (-∞,∞); inverse: D and R both (-∞,∞)

c. D: (-∞,∞), R: (-∞,∞); (no inverse function)

d. D: (-∞,∞), R: (-∞,∞); inverse: D and R both (-∞,∞)

e. D: (-∞,∞), R: [0,∞); (no inverse function)

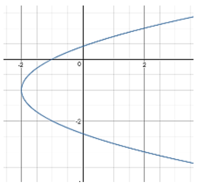
f. D: (-∞,∞), R: (-∞,∞); (no inverse function)

g. D: [0,∞), R: [0,∞); inverse: D: [0,∞), R: [0,∞)

h. D: (-∞,∞), R: (0,∞); inverse: D: (0,∞), R: (-∞,∞)

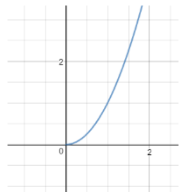
3. b. $f^{-1}(x) = x - 2$ d. $f^{-1}(x) = -\sqrt[3]{x}$ g. $f^{-1}(x) = x^2, x \geq 0$ h. $f^{-1}(x) = log_2 x$

4. a.

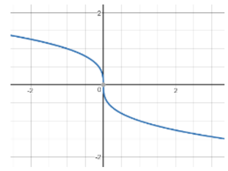


(inverse is graphed here, but it is not a function)

b.



c.



5. Inverse: $y = \frac{x+2}{3}$

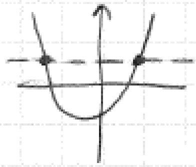
6. a. not inverses

b. inverses

(see following pages for detailed solutions)

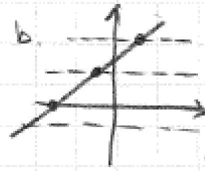
Detailed Solutions

1. a.



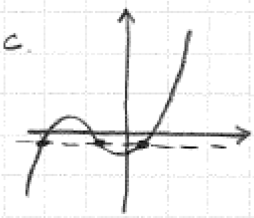
does not pass the horizontal line test (2 different x-values have same y-value) So, <u>not</u> a one-to-one function.

b.



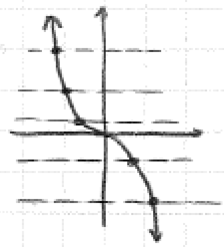
Each horizontal line touches the graph only once, so this <u>is</u> a one-to-one function.

c.



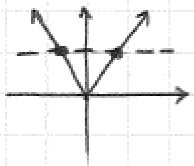
<u>not</u> one-to-one

d.



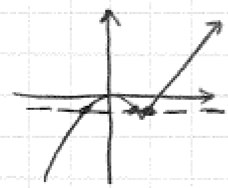
<u>yes</u>, this is a one-to-one function.

e.



<u>not</u> one-to-one

f.

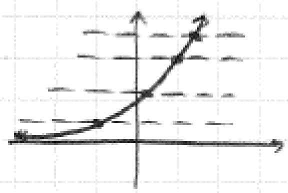


<u>not</u> one-to-one.

g.

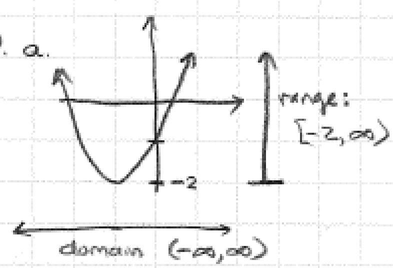
<u>yes</u>, one-to-one

h

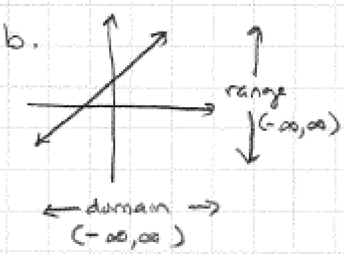


<u>yes</u>, one-to-one

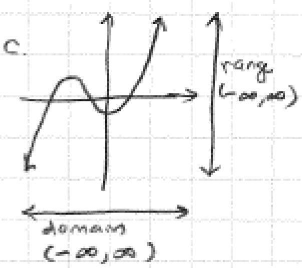
2. a.



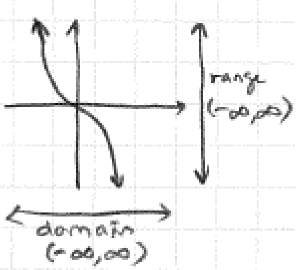
b.



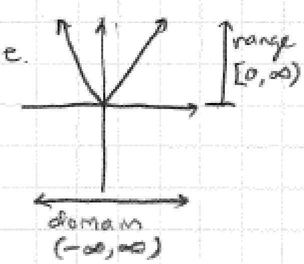
c.



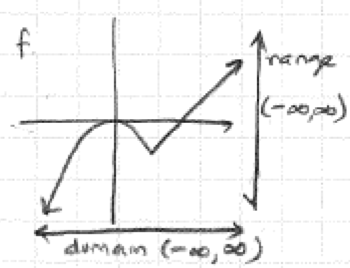
d.



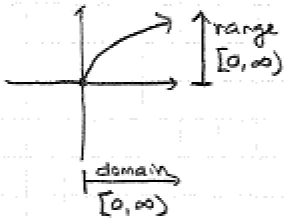
e.



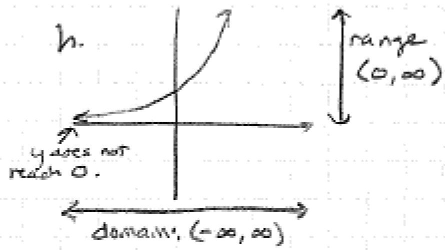
f.



g.



h.



for the functions which have inverses: domain of $f(x)$ = range of $f^{-1}(x)$
and range of $f(x)$ = domain of $f^{-1}(x)$

b. $f(x)$ → $f^{-1}(x)$
D: $(-\infty,\infty)$ → D: $(-\infty,\infty)$
R: $(-\infty,\infty)$ → R: $(-\infty,\infty)$

d. $f(x)$ → $f^{-1}(x)$
D: $(-\infty,\infty)$ → D: $(-\infty,\infty)$
R: $(-\infty,\infty)$ → R: $(-\infty,\infty)$

g. $f(x)$ → $f^{-1}(x)$
D: $[0,\infty)$ → D: $[0,\infty)$
R: $[0,\infty)$ → R: $[0,\infty)$

h. $f(x)$ → $f^{-1}(x)$
D: $(-\infty,\infty)$ → D: $(0,\infty)$
R: $(0,\infty)$ → R: $(-\infty,\infty)$

3. b. $f(x) = y = x+2$

1. reverse variables: $x = y+2$
2. solve for y: $y = x-2$
3. $y \to f^{-1}(x)$: $\boxed{f^{-1}(x) = x-2}$

d. $f(x) = y = -x^3$

1. $x = -y^3$
2. $y^3 = -x$ $\quad y = \sqrt[3]{-x} = -\sqrt[3]{x}$
3. $\boxed{f^{-1}(x) = \sqrt[3]{-x} \text{ or } -\sqrt[3]{x}}$

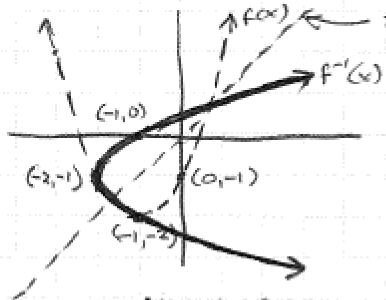
g. $f(x) = \sqrt{x}$

1. $y = \sqrt{x} \to x = \sqrt{y}$
2. $x^2 = y$ (but note that the domain of $y = x^2$ would be $(-\infty,\infty)$ so domain of $f^{-1}(x)$ must be artificially restricted
3. $\boxed{f^{-1}(x) = x^2,\ x \geq 0}$

h. $f(x) = y = 2^x$

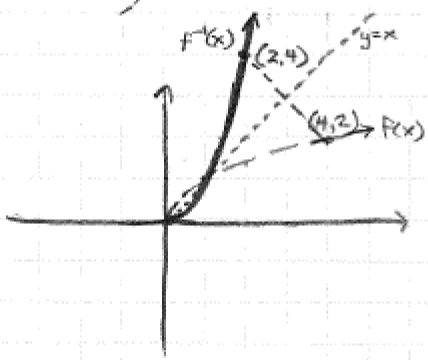
1. $x = 2^y$
(exponential form to log form: $a^b = c \Leftrightarrow \log_a c = b$)
2. $y = \log_2 x$
3. $\boxed{f^{-1}(x) = \log_2 x}$

4. a. $y = (x+1)^2 - 2$ is not a one-to-one function so its inverse is not a function. However, the inverse relation is shown:

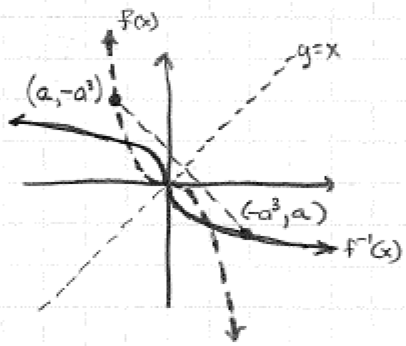


f(x) and $f^{-1}(x)$ are symmetrical to each other with the line $y=x$ as the axis of symmetry.

b.



c.



5. $f(x) = 3x - 2$

$y = 3x - 2$

inverse: $x = 3y - 2$

$3y = x + 2$

$$\boxed{y = \frac{x+2}{3} = f^{-1}(x)}$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = 3f^{-1}(x) - 2$$

$$= \cancel{3}\left[\frac{x+2}{\cancel{3}}\right] - 2 = x + 2 - 2$$

$$\equiv x$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = \frac{f(x)+2}{3}$$

$$= \frac{3x-2+2}{3} = \frac{3x}{3} \equiv x.$$

6. a. are $f(x) = \frac{3}{4}x + 8$ and $g(x) = -\frac{4}{3}x - 8$ inverses?

$$(f \circ g)(x) = \frac{3}{4}g(x) + 8 = \frac{3}{4}\left(-\frac{4}{3}x - 8\right) + 8$$

$$= -x - \frac{24}{4} + 8 = -x - 6 + 8$$

$$= -x + 2 \neq x$$

so, $f(x)$ and $g(x)$ are not inverses.

b. $f(x) = \sqrt[3]{x-6} + 3$ $\qquad g(x) = (x-3)^3 + 6$

$$(f \circ g)(x) = \sqrt[3]{g(x) - 6} + 3 = \sqrt[3]{(x-3)^3 + 6 - 6} + 3$$

$$= \sqrt[3]{(x-3)^3} + 3 = (x-3) + 3$$

$$\equiv x$$

also, $(g \circ f)(x) = (f(x) - 3)^3 + 6$

$$= \left(\sqrt[3]{x-6} + 3 - 3\right)^3 + 6$$

$$= \left(\sqrt[3]{x-6}\right)^3 + 6 = x - 6 + 6 \equiv x$$

Since $(f \circ g)(x) \equiv x$ and $(g \circ f)(x) \equiv x$, $f(x)$ and $g(x)$ are inverses of each other.

Additional Resources

1. Go to http://www.kutasoftware.com/freeia2.html
In the category "General Functions", click on the link for "Inverse Functions"

2. Go to www.khanacademy.org/math/algebra2
Click on "Functions"
On the menu bar on the left, choose from topics on Inverse Functions:
Introduction to Inverses of Functions; Finding Inverse Functions;
Verifying that functions are inverses; Determining whether of function is invertible.