DIY: Trigonometric Identities

Introduction: Identities are equations that are true for every value of the variable(s) in the equation. For example, 3 + 7 = 10 and 3x + 7x = 10x are identities, but 3x + 7x = 10 is not an identity since it is only true if x = 1.

Since all six trigonometric functions are defined by only two independent qualities (the x- and y-coordinates of a point on the unit circle, or the lengths of the opposite and adjacent sides of a right triangle), there is much interdependence in these functions.

An example of a trigonometric identity comes from the definitions of sine and cosine;

$$\sin x = \frac{opposite}{hypotenuse} \text{ and } \cos x = \frac{adjacent}{hypotenuse}$$

so $\frac{\sin x}{\cos x} = \frac{\frac{opposite}{hypotenuse}}{\frac{adjacent}{hypotenuse}} = \frac{opposite}{hypotenuse} \cdot \frac{hypotenuse}{adjacent} = \frac{opposite}{adjacent} = \tan x$ by definition.

So $\tan x = \frac{\sin x}{\cos x}$ is an identity. It is true for any angle x.

To learn about other trigonometric identities, how to prove they are identities, and how to use identities to simplify expressions and evaluate trigonometric quantities, watch the following set of YouTube videos. They are followed by several practice problems for you to try, using the concepts covered in the videos, with answers and detailed solutions. Some additional resources are included for more practice at the end.

1. Co-function identities: https://www.youtube.com/watch?v=MCPK30LNnwU
2. Basic Pythagorean identities:- https://www.youtube.com/watch?v=I4mcja8abDc
3. Sum and difference and double angle identities:-https://www.youtube.com/watch?v=2wcNbjcL5_c
4. Use of sum and difference identities to evaluate trigonometric ratios of certain angles
   a. Example 1:- https://www.youtube.com/watch?v=NZ2Y5_XxzTc&list=PL86281C72D802CE05&index=82
   b. Example2:- https://www.youtube.com/watch?v=c5Zx6Ak8Bt4&list=PL86281C72D802CE05&index=83
   c. Example3:- https://www.youtube.com/watch?v=f-ZshUFbZqk&list=PL86281C72D802CE05&index=84
   d. Example4:- https://www.youtube.com/watch?v=7fy2U0Sm1Vc&list=PL86281C72D802CE05&index=89
5. Half angle identities:- https://www.youtube.com/watch?v=RDH_RiihX6k
6. Power Reducing Identities and Product-to-Sum Identities: https://www.youtube.com/watch?v=otfj1VDFfIs
7. Verifing identities: https://www.youtube.com/watch?v=Qhsf8Ez679Q
8. More on verifying identities: https://www.youtube.com/watch?v=YYKGkYkKeMM
9. Solving Trigonometric equations: https://www.youtube.com/watch?v=IE0FxGegdMg
10. Solving trigonometric equations: https://www.youtube.com/watch?v=qOboetZIiM4

Summary of Trigonometric Identities

1. Fundamental Identities:

- Cofunction Identities: $\sin x = \cos(90^o - x)$ $\quad \cos x = \sin(90° - x)$
  $\tan x = \cot(90° - x)$ $\quad \cot x = \tan(90° - x)$
  $\sec x = csc(90° - x)$ $\quad \csc x = \sec(90° - x)$
  (Note: substitute π/2 for 90^0 if using radian measure)

- Quotient Identities: $\tan x = \frac{\sin x}{\cos x}$ $\quad \cot x = \frac{\cos x}{\sin x}$

- Reciprocal Identities: $\cot x = \frac{1}{\tan x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \csc x = \frac{1}{\sin x}$

- Even/Odd functions: $\sin(-x) = -\sin x$ (odd function
  $\cos(-x) = \cos x$ (even function)
  $\tan(-x) = -\tan x$ (odd function)

- Pythagorean Identities: $\cos^2 x + \sin^2 x = 1$ $\quad 1 + \tan^2 x = \sec^2 x$ $\quad 1 + \cot^2 x = \csc^2 x$

2. Sum Identities:

$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp tan\, x\, tany}$

3. Double Angle and Power Reducing Identities:

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

$\sin^2 x = \frac{1}{2}(1 - \cos\ 2x)$ $\quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

4. Half-Angle Identities:

$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$ $\quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

5. Product-to-Sum and Sum-to-Product Identities:

- Product-to-Sum:
  $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$
  $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
  $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

- Sum-to-Product:
  $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ $\quad \sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$
  $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ $\quad \cos x + \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

Practice problems: The following problems use the techniques demonstrated in the above videos. The

1) Verify the following identities:

a) $\sin^4 x - \cos^4 x = (\sin x - \cos x)(\sin x + \cos x)$

b) $\sec^2 A - 1 = \tan^2 A$

c) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$

d) $\frac{1+\tan^2\beta}{1+\cot^2\beta} = \tan^2\beta$

2) Evaluate the following without using a calculator. Write the exact answer:

a) $\sin^2 117° + \cos^2 117°$

b) $\cos(-135°)$

c) $\sin 100° \cos 10° - \sin 10° \cos 100°$

d) $\frac{\tan\frac{5\pi}{12} + \tan\frac{\pi}{4}}{1 - \tan\frac{5\pi}{12}\tan\frac{\pi}{4}}$

e) $-\sin 150° \sin 60° + \cos 150° \cos 60°$

f) $\sin 75°$

g) $\frac{2\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{1-2\sin^2\frac{\pi}{6}}$

h) $\frac{1}{\sec^2(\frac{\pi}{4})-1}$

i) $\cos^2 15° - \sin^2 15°$

j) $\sin^2 29° + 2\sin 60° \cos 60° + \cos^2 29°$

3) Find the exact values for the following, using identities and values of trig functions for special angles (such as 30^0, 45^0, 60^0, 90^0, etc.)

a) $\sin 105^0$

b) $\tan 225^0$

c) $\cos 165^0$

4) Verify the following identities: (Note: these are more challenging than the identities in problem 1, involving using identities and algebraic techniques.)

a) $\frac{1-\tan A}{1+\tan A} = \frac{\cot A - 1}{\cot A + 1}$

b) $\cos^6\theta + \sin^6\theta = 1 - 3\sin^2\theta\cos^2\theta$
(hint: think of the left-hand side as the sum of cubes, then factor)

c) $\sec 2x = \frac{\sec^2 x + \sec^4 x}{2 + \sec^2 x - \sec^4 x}$

d) $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$

e) $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1$

f) $\sin B = \dfrac{2\tan\frac{\mathbf{B}}{\mathbf{2}}}{1+\tan^2\frac{\mathbf{B}}{\mathbf{2}}}$
(hint: use the identity from part d)

g) $\cos 2t = \dfrac{\cot t - \tan t}{\csc t \sec t}$

h) $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

i) $(\sin \alpha + \csc \alpha)^2 + (\cos \alpha + \sec \alpha)^2 = \tan^2\alpha + \cot^2\alpha + 7$

5) Evaluate:

a) If $\sin x = \frac{12}{13}$, find tan x and cos x

b) If $\cos^2 A = \frac{3}{4}$, find cot A and csc A

c) Find sin (s+t) and cos (s+t) given that sin s = -4/5 and cos t = 12/13. Angle s is in Q III and angle t is in Q IV. What quadrant is (s+t) in?

6) Find all solutions in $[0,2\pi)$
: *Note: these are NOT identities. They are only true for particular values of the variable.*

a) $\sin^2\theta + 2\cos\theta - 2 = 0$

b) $\cot x + \tan x = 2\csc x$

c) $\cot^2 B + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot B + 1 = 0$

d) $\sin 5\theta = \cos 5\theta$

e) $\tan\alpha + \sec\alpha = \sqrt{3}$

f) $\csc x = 1 + \cot x$

7) Prove the following:

a) $\sec^2\frac{x}{2} = \dfrac{2}{1+\cos x}$

b) $-\cot\frac{x}{2} = \dfrac{\sin 2x + \sin x}{\cos 2x - \cos x}$

c) $\dfrac{\sin 3t + \sin 2t}{\sin 3t - \sin 2t} = \dfrac{\tan\frac{5t}{2}}{\tan\frac{t}{2}}$

Hint: Use Sum-to-Product Identities

d) $\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \sec x + \tan x$ *Hint: Use the sum identity for tangent, then convert all terms to sines and cosines.*

Answers:

1. (Proofs)

2. a. 1 b. $\frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$ c. 1 d. $-\sqrt{3}$ e. $\frac{-\sqrt{3}}{2}$ f. $\frac{\sqrt{2}+\sqrt{6}}{4}$ g. $\sqrt{3}$ h. 1

   i. $\frac{\sqrt{3}}{2}$ j. $1 + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$

3. a. $\frac{\sqrt{6}+\sqrt{2}}{4}$ b. 1 c. $-\frac{\sqrt{6}+\sqrt{2}}{4}$

4. (Proofs)

5. a. $\cos x = \pm\frac{5}{13}$, $\tan x = \pm\frac{12}{5}$ b. $\cot A = \pm\sqrt{3}$, $\csc A = \pm\sqrt{2}$

   c. $\sin(s+t) = \frac{-33}{65}$, $\cos(s+t) = \frac{-56}{65}$, (s+t) is in Quadrant III

6. a. 0 b. $\frac{\pi}{3}, \frac{5\pi}{3}$ c. $\frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{6}, \frac{11\pi}{6}$ d. $\frac{\pi}{20}, \frac{9\pi}{20}, \frac{17\pi}{20}, \frac{25\pi}{20} = \frac{5\pi}{4}, \frac{33\pi}{20}, \frac{5\pi}{20} = \frac{\pi}{4}, \frac{13\pi}{20}, \frac{21\pi}{20}, \frac{29\pi}{20}, \frac{37\pi}{20}$

   e. $\frac{\pi}{6}$ f. $\frac{\pi}{2}$

7. (Proofs)

Detailed Solutions

1.a. Verify: $\sin^4 x - \cos^4 x = (\sin x - \cos x)(\sin x + \cos x)$

* LHS = left-hand side
RHS = right-hand side.

factoring the LHS: $\sin^4 x - \cos^4 x$

$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$

since $\sin^2 x + \cos^2 x = 1$ $\quad = \sin^2 x - \cos^2 x$

$= (\sin x - \cos x)(\sin x + \cos x) = \text{RHS}.$

b. $\sec^2 A - 1 = \tan^2 A$

using Pythagorean ID $\tan^2 A + 1 = \sec^2 A$

LHS: $\tan^2 A + 1 - 1$

$= \tan^2 A$

$= \text{RHS}.$

note: It might seem easier to add 1 to both sides of this to make the Pythagorean Identity $\sec^2 A = \tan^2 A + 1$, but when verifying identities, must work with each side independently of the other side.

c. $$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$$

LHS: often when the denominator of a term is a binomial, multiplying numerator and denominator by the conjugate of the denominator helps.

$$\text{LHS} = \frac{\sin\theta}{(1+\cos\theta)} \cdot \frac{(1-\cos\theta)}{(1-\cos\theta)} + \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{\sin\theta\,(1-\cos\theta)}{1-\cos^2\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta\,(1-\cos\theta)}{\sin^2\theta} + \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{\cancel{\sin\theta}\,(1-\cos\theta)}{\cancel{\sin\theta}\cdot\sin\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{1-\cos\theta+1+\cos\theta}{\sin\theta} = \frac{2}{\sin\theta}$$

$$= 2\csc\theta = \text{RHS}$$

1. d. $\dfrac{1+\tan^2\beta}{1+\cot^2\beta} = \tan^2\beta$

using Pythagorean identities: $LHS = \dfrac{\sec^2\beta}{\csc^2\beta}$

$$= \sec^2\beta \cdot \frac{1}{\csc^2\beta}$$

$$= \frac{1}{\cos^2\beta} \cdot \sin^2\beta = \frac{\sin^2\beta}{\cos^2\beta} = \tan^2\beta$$

2. a. $\sin^2 117^\circ + \cos^2 117^\circ = 1$ This is a Pythagorean identity. As long as the angle in both terms are equal, $\sin^2\theta + \cos^2\theta = 1$.

b. $\cos(-135^\circ) = \cos 135^\circ$ (negative angle identity)

$$= \cos(90^\circ + 45^\circ) = \cos 90^\circ \cos 45^\circ - \sin 90^\circ \sin 45^\circ$$
$$= (0)(\sqrt{2}/2) - (1)(\sqrt{2}/2)$$
$$= \boxed{-\sqrt{2}/2}$$

or $\cos 135^\circ = \sin(90^\circ - 135^\circ) = \sin(-45^\circ) = -\sin 45^\circ = \boxed{\dfrac{-\sqrt{2}}{2}}$

c. $\sin 100^\circ \cos 10^\circ - \sin 10^\circ \cos 100^\circ = \sin(100^\circ - 10^\circ)$ (sum identity for sine)

$$= \sin 90^\circ$$
$$= 1$$

d. $\dfrac{\tan\frac{5\pi}{12} + \tan\frac{\pi}{4}}{1 - \tan\frac{5\pi}{12}\tan\frac{\pi}{4}} = \tan\left(\dfrac{5\pi}{12} + \dfrac{\pi}{4}\right)$

$$= \tan\left(\frac{5\pi}{12} + \frac{3\pi}{12}\right) = \tan\frac{8\pi}{12}$$

$$= \tan\frac{2\pi}{3} \quad (Q.\,II)$$

(use reference angle for Q. II $\theta' = \pi - \theta$) $= -\tan\left(\pi - \dfrac{2\pi}{3}\right) = -\tan\dfrac{\pi}{3} = \boxed{-\sqrt{3}}$

2. e. $-\sin 150^\circ \sin 60^\circ + \cos 150^\circ \cos 60^\circ$

$= \cos 150^\circ \cos 60^\circ - \sin 150^\circ \sin 60^\circ$

$= \cos(150^\circ + 60^\circ) = \cos 210^\circ$ Q. III

$= -\cos(210^\circ - 180^\circ) \leftarrow$ using reference angle for Q. III: $\theta' = \theta - 180^\circ$

$= -\cos 30^\circ$

$= \boxed{-\frac{\sqrt{3}}{2}}$

or using only identities: $\cos(210^\circ) = \cos(180^\circ + 30^\circ)$

$= \cos 180^\circ \cos 30^\circ - \sin 180^\circ \sin 30^\circ$

$= (-1)(\sqrt{3}/2) - (0)(1/2)$

$= -\frac{\sqrt{3}}{2}$

f. $\sin 75^\circ = \sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

g. $\dfrac{2\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{1 - 2\sin^2\frac{\pi}{6}}$

Evaluating directly: $= \dfrac{2(1/2)(\sqrt{3}/2)}{1 - 2(1/2)^2}$

$= \dfrac{\sqrt{3}/2}{1 - 1/2} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

using identities: $= \dfrac{\sin(2 \cdot \frac{\pi}{6})}{\cos(2 \cdot \frac{\pi}{6})}$

$= \dfrac{\sin \pi/3}{\cos \pi/3} = \tan\frac{\pi}{3} = \boxed{\sqrt{3}}$

h. $\dfrac{1}{\sec^2(\pi/4) - 1} = \dfrac{1}{\tan^2\frac{\pi}{4} + 1 - 1} = \dfrac{1}{\tan^2\frac{\pi}{4}} = \dfrac{1}{(1)^2} = 1$

or using $\sec \pi/4 = \sqrt{2}$ $\quad \dfrac{1}{(\sqrt{2})^2 - 1} = \dfrac{1}{2-1} = 1$

2.i. $\cos^2 15° - \sin^2 15° = \cos(2 \cdot 15°) = \cos 30° = \frac{\sqrt{3}}{2}$

j. $\sin^2 29° + 2\sin 60° \cos 60° + \cos^2 29°$

$= \sin^2 29° + \cos^2 29° + 2\sin 60° \cos 60°$

$= 1 + \sin(2 \cdot 60°)$

$= 1 + \sin 120° = 1 + \sin(180° - 120°) = 1 + \sin 60°$

$= \boxed{1 + \frac{\sqrt{3}}{2}}$ or $\boxed{\frac{2+\sqrt{3}}{2}}$

3.a) $\sin 105° = \sin(90° + 15°)$ but this doesn't help since $\sin 15°$, $\cos 15°$ isn't known

$\sin 105° = \sin(60° + 45°)$

$= \sin 60° \cos 45° + \sin 45° \cos 60°$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

b) $\tan 225° = \tan(180° + 45°) = \frac{\tan 180° + \tan 45°}{1 - \tan 180° \tan 45°}$

$= \frac{0+1}{1-0(1)} = \frac{1}{1} = 1$

c) $\cos 165° = \cos(120° + 45°) = \cos 120° \cos 45° - \sin 120° \sin 45°$

$= (-1/2)(\sqrt{2}/2) - (\sqrt{3}/2)(\sqrt{2}/2)$

$= -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \boxed{\frac{-(\sqrt{2}+\sqrt{6})}{4}}$

4.a. $\dfrac{1-\tan A}{1+\tan A} = \dfrac{\cot A - 1}{\cot A + 1}$

LHS: $= \dfrac{1-\frac{\sin A}{\cos A}}{1+\frac{\sin A}{\cos A}}$

$= \dfrac{\left(1-\frac{\sin A}{\cos A}\right)\cos A}{\left(1+\frac{\sin A}{\cos A}\right)\cos A}$

$= \dfrac{\cos A - \sin A}{\cos A + \sin A}$

RHS: $= \dfrac{\frac{\cos A}{\sin A}-1}{\frac{\cos A}{\sin A}+1}$

$= \dfrac{\left(\frac{\cos A}{\sin A}-1\right)\sin A}{\left(\frac{\cos A}{\sin A}+1\right)\sin A}$

$= \dfrac{\cos A - \sin A}{\cos A + \sin A}$

LHS = RHS.

b. $\cos^6\theta + \sin^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

LHS $= (\cos^2\theta)^3 + (\sin^2\theta)^3$ use $a^3+b^3=(a+b)(a^2-ab+b^2)$

RHS $= 1-3(1-\cos^2\theta)\cos^2\theta$

$= 1-3\cos^2\theta + 3\cos^4\theta$

$= (\cos^2\theta+\sin^2\theta)(\cos^4\theta - \sin^2\theta\cos^2\theta + \sin^4\theta)$

$= (1)(\cos^4\theta - \sin^2\theta\cos^2\theta + \sin^4\theta)$

$= \cos^4\theta - (1-\cos^2\theta)\cos^2\theta + (1-\cos^2\theta)^2$

$= \cos^4\theta - \cos^2\theta + \cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 3\cos^4\theta + 1 - \cos^2\theta - 2\cos^2\theta$

$= 3\cos^4\theta + 1 - 3\cos^2\theta$ = RHS. but can go on...

$= 1 + 3\cos^2\theta(\cos^2\theta - 1) = 1 + 3\cos^2\theta(-1)(1-\cos^2\theta)$

$= 1 + 3\cos^2\theta(-\sin^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta$

= RHS (original).

4.c. $\sec 2x = \dfrac{\sec^2 x + \sec^4 x}{2 + \sec^2 x - \sec^4 x}$

$$RHS = \frac{\sec^2 x\,(1+\sec^2 x)}{(2-\sec^2 x)(1+\sec^2 x)}$$

$$= \frac{\sec^2 x}{2-\sec^2 x} = \frac{\frac{1}{\cos^2 x}}{2-\frac{1}{\cos^2 x}}$$

$$LHS = \sec 2x = \frac{1}{\cos 2x}$$

$$= \frac{1}{2\cos^2 x - 1}$$

$$= \frac{\cos^2 x\left[\frac{1}{\cos^2 x}\right]}{\cos^2 x\left[2-\frac{1}{\cos^2 x}\right]}$$

$$RHS = \frac{1}{2\cos^2 x - 1} = LHS$$

d. $\tan\left(\frac{A}{2}\right) = \dfrac{1-\cos A}{\sin A}$

$$LHS = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \frac{\pm\sqrt{\frac{1-\cos A}{2}}}{\pm\sqrt{\frac{1+\cos A}{2}}}$$

$$= \pm\frac{\sqrt{1-\cos A}}{\sqrt{1+\cos A}} = \pm\frac{\sqrt{1-\cos A}}{\sqrt{1+\cos A}}\cdot\frac{\sqrt{1-\cos A}}{\sqrt{1-\cos A}} = \pm\frac{\left(\sqrt{1-\cos A}\right)^2}{\sqrt{1-\cos^2 A}}$$

$$= \pm\frac{1-\cos A}{\sqrt{\sin^2 A}} = \pm\frac{1-\cos A}{\sin A} = RHS.$$

e. $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ) = 1$

$LHS = \sin(420^\circ - 360^\circ)\cos(390^\circ - 360^\circ) + \cos(-300^\circ + 360^\circ)\sin(-330^\circ + 360^\circ)$ *

$$LHS = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$$
$$= \sin(60^\circ + 30^\circ) = \sin 90^\circ = 1 = RHS$$

* this uses properties of coterminal angles. $\sin\theta = \sin(\theta \pm n\cdot 360^\circ)$ (true for all 6 trig. functions)

4.f. $\sin B = \dfrac{2\tan\left(\frac{B}{2}\right)}{1+\tan^2\frac{B}{2}}$ using $\tan\left(\frac{B}{2}\right) = \dfrac{1-\cos B}{\sin B}$

$$RHS = \frac{2\left[\frac{1-\cos B}{\sin B}\right]}{1+\left(\frac{1-\cos B}{\sin B}\right)^2} = \frac{\frac{2(1-\cos B)}{\sin B}\cdot \sin^2 B}{\left[1+\frac{(1-\cos B)^2}{\sin^2 B}\right]\sin^2 B}$$

$$= \frac{2\sin B(1-\cos B)}{\sin^2 B + (1-\cos B)^2} = \frac{2\sin B(1-\cos B)}{\underbrace{\sin^2 B + 1 - 2\cos B + \cos^2 B}_{=1}}$$

$$= \frac{2\sin B(1-\cos B)}{1+1-2\cos B} = \frac{2\sin B(1-\cos B)}{2-2\cos B}$$

$$= \frac{\cancel{2}\sin B\,\cancel{(1-\cos B)}}{\cancel{2}\cancel{(1-\cos B)}} = \sin B = LHS$$

g. $\cos 2t = \dfrac{\cot t - \tan t}{\csc t\ \sec t}$

With so many different functions, it would be worth beginning by converting all to sines and cosines.

$$RHS = \frac{\frac{\cos t}{\sin t} - \frac{\sin t}{\cos t}}{\frac{1}{\sin t}\cdot\frac{1}{\cos t}}$$

Multiply numerator and denominator by the common denominator $\sin t \cos t$

$$= \frac{\sin t\cos t\left(\frac{\cos t}{\sin t} - \frac{\sin t}{\cos t}\right)}{\cancel{\sin t\cos t}\left(\frac{1}{\cancel{\sin t\cos t}}\right)} = \frac{\cos^2 t - \sin^2 t}{1}$$

$$= \cos^2 t - \sin^2 t = \cos 2t = LHS.$$

4.h) $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

$$LHS = \sqrt{\frac{1-\sin A}{1+\sin A}} \cdot \sqrt{\frac{1-\sin A}{1-\sin A}}$$

$$= \sqrt{\frac{(1-\sin A)^2}{1-\sin^2 A}} = \frac{\sqrt{(1-\sin A)^2}}{\sqrt{\cos^2 A}}$$

$$= \frac{1-\sin A}{\cos A} = \frac{1}{\cos A} - \frac{\sin A}{\cos A} = \sec A - \tan A = RHS.$$

i) $(\sin\alpha + \csc\alpha)^2 + (\cos\alpha + \sec\alpha)^2 = \tan^2\alpha + \cot^2\alpha + 7$

$$LHS = \sin^2\alpha + 2\sin\alpha\csc\alpha + \csc^2\alpha + \cos^2\alpha + 2\cos\alpha\sec\alpha + \sec^2\alpha$$

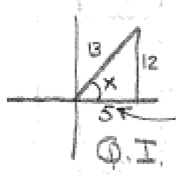
$$= \sin^2\alpha + \cos^2\alpha + 2\sin\alpha \cdot \frac{1}{\sin\alpha} + 2\cos\alpha \cdot \frac{1}{\cos\alpha} + \csc^2\alpha + \sec^2\alpha$$

$$= 1 + 2 + 2 + \underline{\csc^2\alpha} + \sec^2\alpha$$

$$= 5 + \cot^2\alpha + 1 + \tan^2\alpha + 1 = 7 + \tan^2\alpha + \cot^2\alpha = RHS.$$

5. a. Evaluate $\tan x$ and $\cos x$ if $\sin x = \frac{12}{13} = \frac{y}{r}$

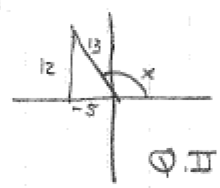
(x could be in Q. I or II, so both $\cos x$ and $\tan x$ could be $\pm$)



$= \sqrt{13^2 - 12^2}$
$= \sqrt{169-144}$
$= \sqrt{25} = 5$

$\cos x = \frac{5}{13}, \; \frac{-5}{13}$

$\tan x = \frac{12}{5}, \; \frac{-12}{5}$

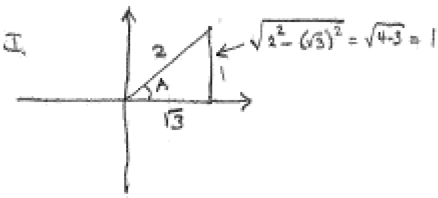


5.b. If $\cos^2 A = \frac{3}{4}$, find $\cot A$ and $\csc A$

$\cos A = \pm \frac{\sqrt{3}}{2}$ A could be in any quadrant, so $\cot A$, $\csc A$ could be $\pm$

$= \frac{x}{r}$

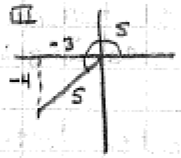
Q. I

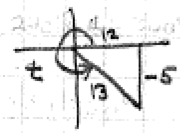


$\cot A = \frac{x}{y} = \pm \frac{\sqrt{3}}{1} = \pm \sqrt{3}$

$\csc A = \frac{r}{y} = \pm \frac{2}{1} = \pm 2$

c. $\sin s = -\frac{4}{5}$ $\cos t = \frac{12}{13}$ s in Q. III t in Q. IV





(missing values found using Pythagorean Theorem)

$\sin(s+t) = \sin s \cos t + \sin t \cos s$

$= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{-5}{13}\right)\left(\frac{-3}{5}\right)$

$= -\frac{48}{65} + \frac{15}{65} = \boxed{-\frac{33}{65} = \sin(s+t)}$

$\cos(s+t) = \cos s \cos t - \sin s \sin t$

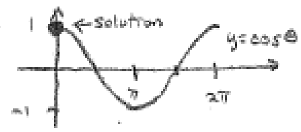
$= \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{-5}{13}\right) = -\frac{36}{65} - \frac{20}{65} = \boxed{-\frac{56}{65} = \cos(s+t)}$

Since both $\sin(s+t)$ and $\cos(s+t)$ are negative, $(s+t)$ must be in $\boxed{\text{Q. III.}}$

6.a. Find all values of θ in $[0, 2\pi)$ that satisfy the equation

$$\sin^2\theta + 2\cos\theta - 2 = 0$$
$$= 1 - \cos^2\theta + 2\cos\theta - 2 = 0$$
$$-\cos^2\theta + 2\cos\theta - 1 = 0$$
$$\cos^2\theta - 2\cos\theta + 1 = 0$$
$$= (\cos\theta - 1)(\cos\theta - 1) = 0$$
$$\text{or} \quad (\cos\theta - 1)^2 = 0$$
$$\cos\theta - 1 = 0$$
$$\cos\theta = 1 \longrightarrow \boxed{\theta = 0}$$

(this is the only angle in $[0, 2\pi)$ for which $\cos\theta = 1$)



b. $\cot x + \tan x = 2\csc x$ ← write all in terms of sin, cos

$$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{2}{\sin x}$$ ← multiply both sides of equation by $\sin x \cdot \cos x$

$$\cos^2 x + \sin^2 x = 2\cos x$$
$$1 = 2\cos x \longrightarrow \cos x = \frac{1}{2}$$ (x could be Q.I, IV)

$$x = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3} = \boxed{\frac{\pi}{3}, \frac{5\pi}{3}}$$

c. $\cot^2 B + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot B + 1 = 0$

using the quadratic formula $\cot B = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\cot B = \frac{-\frac{4}{\sqrt{3}} \pm \sqrt{\frac{16}{3} - 4(1)(1)}}{2(1)}$$

where $a = 1$ $b = \sqrt{3} + \frac{1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$ $c = 1$

$$= \frac{-\frac{4}{\sqrt{3}} \pm \sqrt{\frac{4}{3}}}{2} = \frac{-\frac{4}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}}{2} = -\frac{1}{\sqrt{3}},\ \frac{-3}{\sqrt{3}} = -\sqrt{3}$$ (B must be in Q. II, IV)

$\cot B = -\frac{1}{\sqrt{3}}$ $B = 180° - 60°$ or $360° - 60°$ $= \boxed{120° \text{ or } 300°} = \boxed{\frac{2\pi}{3}, \frac{5\pi}{3}}$

$\cot B = -\sqrt{3}$ $B = 180° - 30° = 150°$ or $360° - 30° = 330°$ $= \boxed{\frac{5\pi}{6}, \frac{11\pi}{6}}$

6.d. $\sin 5\theta = \cos 5\theta$

For $\cos 5\theta \neq 0$, $\dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{\cos 5\theta}{\cos 5\theta}$

$$\tan 5\theta = 1$$

Note: If we want all values of θ in $[0, 2\pi)$ we must consider 5θ in $[0, 10\pi)$.

If $\tan 5\theta = 1$, then 5θ could be $\frac{\pi}{4}$, $\frac{5\pi}{4}$, or any of their coterminal angles in the $[0, 10\pi)$ interval.

$5\theta = \frac{\pi}{4} + n(2\pi) \rightarrow \theta = \frac{\pi}{20} + n\left(\frac{2}{5}\pi\right) = \frac{\pi + n(8\pi)}{20}$ $n = 1-4$

or $\frac{5\pi}{4} + n(2\pi) \rightarrow$ or $\theta = \frac{\pi}{4} + n\left(\frac{2}{5}\pi\right) = \frac{5\pi + n(8\pi)}{20}$ $n = 1-4$

$$\theta = \boxed{\frac{\pi}{20}, \frac{9\pi}{20}, \frac{17\pi}{20}, \underbrace{\frac{25\pi}{20}}_{=\frac{5\pi}{4}}, \frac{33\pi}{20}, \frac{\pi}{4} = \frac{5\pi}{20}, \frac{13\pi}{20}, \frac{21\pi}{20}, \frac{29\pi}{20}, \frac{37\pi}{20}}$$

e. $\tan\alpha + \sec\alpha = \sqrt{3}$

$\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{1}{\cos\alpha} = \sqrt{3}$ multiply by $\cos\alpha$ on both sides.

$\sin\alpha + 1 = \sqrt{3}\cos\alpha$ Square both sides.

$(\sin\alpha + 1)^2 = (\sqrt{3}\cos\alpha)^2$

$\sin^2\alpha + 2\sin\alpha + 1 = 3\cos^2\alpha$

$= 3(1 - \sin^2\alpha)$

$= 3 - 3\sin^2\alpha$

$4\sin^2\alpha + 2\sin\alpha - 2 = 0$

$2(2\sin^2\alpha + \sin\alpha - 1) = 0$

$2(2\sin\alpha - 1)(\sin\alpha + 1) = 0$

$2\sin\alpha - 1 = 0$ $\quad$ $\sin\alpha + 1 = 0$

$\sin\alpha = 1/2$ $\quad$ $\sin\alpha = -1$

$\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad$ $\alpha = \frac{3\pi}{2}$

However, since these are solutions to the squared equation, they must be checked in the original equation.*

* Checking.

1. $\alpha = \frac{\pi}{6}$ $\quad \tan\frac{\pi}{6} + \sec\frac{\pi}{6} = \frac{\sqrt{3}}{3} + \frac{2}{3}\sqrt{3} = \sqrt{3}$ ✓
2. $\alpha = \frac{5\pi}{6}$ $\quad \tan\frac{5\pi}{6} + \sec\frac{5\pi}{6} = -\sqrt{3}$ (not valid)
3. $\alpha = \frac{3\pi}{2}$ $\quad$ Neither tan nor sec are defined for $3\pi/2$ so this is not a valid solution.

Ans: $\boxed{\pi/6}$

6.f. $\csc x = 1 + \cot x$

Squaring, $\csc^2 x = (1+\cot x)^2$

$$= 1 + 2\cot x + \cot^2 x$$

$$\cancel{\cot^2 x + 1} = \cancel{1} + 2\cot x + \cancel{\cot^2 x}$$

$$-\cot^2 x - 1 \qquad -1 \qquad -\cot^2 x$$

$$2\cot x = 0$$

$$\cot x = 0$$

$$x = \frac{\pi}{2},\ \frac{3\pi}{2}$$

Again, we must check these possible solutions in the original equation.

1. $x = \frac{\pi}{2}$ $\qquad \csc\frac{\pi}{2} = 1 + \cot\frac{\pi}{2}$

$$1 = 1 + 0 \quad ✓$$

2. $x = \frac{3\pi}{2}$ $\qquad \csc\frac{3\pi}{2} = 1 + \cot\frac{3\pi}{2}$

$$-1 = 1 + 0 \quad \text{(not valid)}$$

Answer: $\boxed{x = \frac{\pi}{2}}$

7. (back to identities.)

a. $\sec^2\frac{x}{2} = \frac{2}{1+\cos x}$

$$\text{LHS} = \sec^2\frac{x}{2} = \frac{1}{\cos^2\frac{x}{2}}$$

$$= \frac{1}{\left(\sqrt{\frac{1+\cos x}{2}}\right)^2} = \frac{1}{\frac{(1+\cos x)}{2}} \cdot \frac{2}{2}$$

$$= \frac{2}{1+\cos x} = \text{RHS}.$$

7. b. $-\cot\frac{x}{2} = \dfrac{\sin 2x + \sin x}{\cos 2x - \cos x}$

LHS: $-\cot\frac{x}{2} = \dfrac{-1}{\tan\frac{x}{2}}$

using identity proven in prob. 4d:

$$\frac{-1}{\tan\frac{x}{2}} = \frac{-1}{\frac{1-\cos x}{\sin x}} = \frac{-\sin x}{1-\cos x}$$

RHS: $= \dfrac{2\sin x\cos x + \sin x}{2\cos^2 x - 1 - \cos x}$

$$= \frac{\sin x\,(\cancel{2\cos x+1})}{(\cancel{2\cos x + 1})(\cos x - 1)}$$

$$= \frac{\sin x}{\cos x - 1} = \frac{-\sin x}{1-\cos x}$$

LHS = RHS.

c. $\dfrac{\sin 3t + \sin 2t}{\sin 3t - \sin 2t} = \dfrac{\tan\frac{5t}{2}}{\tan\frac{t}{2}}$

using: $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

and also then: $\sin x + \sin(-y) = \sin x - \sin y = 2\sin\left(\frac{x+(-y)}{2}\right)\cos\left(\frac{x-(-y)}{2}\right)$

so $\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

$$\text{LHS} = \frac{\sin 3t + \sin 2t}{\sin 3t - \sin 2t} = \frac{\cancel{2}\sin\left(\frac{3t+2t}{2}\right)\cos\left(\frac{3t-2t}{2}\right)}{\cancel{2}\sin\left(\frac{3t-2t}{2}\right)\cos\left(\frac{3t+2t}{2}\right)}$$

$$= \frac{\sin(5t/2)\cos(t/2)}{\sin(t/2)\cos(5t/2)} = \frac{\sin(5t/2)}{\cos(5t/2)}\cdot\frac{\cos(t/2)}{\sin(t/2)}$$

$$= \tan(5t/2)\cdot\cot(t/2) = \frac{\tan(5t/2)}{\tan(t/2)} = \text{RHS}.$$

7.d. $\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) = \sec x + \tan x$

$$LHS = \frac{\tan\frac{x}{2}+\tan\frac{\pi}{4}}{1-\tan\frac{x}{2}\tan\frac{\pi}{4}}$$

$$= \frac{\tan\frac{x}{2}+1}{1-\tan(x/2)(1)}$$

Using $\tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$

Now, break $\tan\frac{x}{2}$ into $\frac{\sin x/2}{\cos x/2}$

$$= \frac{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+1}{1-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}$$

Multiply numerator & denominator by $\cos\frac{x}{2}$

$$= \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\cdot\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$= \frac{\sin\frac{x}{2}\cos\frac{x}{2}+\overbrace{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}^{=1}+\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}+1}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{\sin 2\left(\frac{x}{2}\right)+1}{\cos 2\left(\frac{x}{2}\right)}$$

$$= \frac{\sin x+1}{\cos x} = \frac{\sin x}{\cos x}+\frac{1}{\cos x} = \tan x + \sec x = RHS.$$

<u>Additional Resources</u>

1. Go To https://www.kutasoftware.com/freeipc.html

2. Under "**Trigonometry**" click on:
   a) Simple trig equations
   b) Simple trig equations
   c) Inverse trig functions
   d) Fundamental identities
   e) Equations with factoring and fundamental identities
   f) Sum and Difference Identities
   g) Multiple-Angle Identities
   h) Product-to-Sum Identities
   i) Equations and Multiple-Angle Identities

3. You can print out the worksheets and work on them.

4. For help please contact the ***Math Assistance Area***.