# DIY: Gaussian and Gauss-Jordan Elimination

To review these matrix methods for solving systems of linear equations, watch the following set of YouTube videos**.** They are followed by several practice problems for you to try, covering all the basic concepts covered in the videos, with answers and detailed solutions. Some additional resources are included for more practice at the end.

1. Part 1: Explanation of Gaussian elimination
2. Part 2: Gaussian elimination using matrices

**Note of Explanation:** The difference between Gaussian elimination using matrices and the Gauss-Jordan elimination method is in where the matrix manipulation stops. (In some videos, this difference is not made clear, or the wrong name is attached to the method being used.) So, to clarify…

**Gaussian elimination** (used in the video above) or “Gaussian reduction”, stops when the augmented coefficient matrix is in “upper-triangular” or “row-echelon” form, such as the matrix shown here: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right]$. From here, the final solution is found using back-substitution.

**Gauss-Jordan elimination** ( used in the videos below) stops when the augmented coefficient matrix is in “diagonal” or “reduced row echelon” form, such as this matrix; $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right]$. From here, the final solution can be immediately seen without back-substitution.

3. Solving a 2 x 2 System using Gauss-Jordan elimination
4. Solving a 3 x 3 system using Gauss-Jordan elimination
5. Solving a 3x3 system using Gauss-Jordan-- another presentation *
6. Dependent and Inconsistent systems of equations

*Note: the presenter in this video states that is not correct to divide a row by a number but rather you must multiply by its reciprocal. This is not necessary. The two operations, dividing by a (non-zero) constant and multiplying by its reciprocal are equivalent.

## Practice Problems:

1. a. Solve using Gaussian elimination with matrices:

$$x + 2y = 1$$
$$2x - y = 7$$

   b. Solve the system in 1.a. using Gauss-Jordan elimination.

2. Solve using Gauss-Jordan elimination: $3x + 4y = 4$, $6x - 2y = 3$

$$3x + 4y = 4$$
$$6x - 2y = 3$$

3. The following matrices represent systems of 3 equations with 3 variables. Gauss-Jordan elimination was used to arrive at the given matrices. Express the solution indicated by each matrix in the form (x, y, z) or state that no solution exists.

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ c. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ e. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$ f. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

4. Solve the following using Gauss-Jordan elimination:

$$-x + y = -1$$
$$y - z = 6$$
$$x + z = -1$$

5. Solve the following using Gauss-Jordan elimination:

$$x + 2y - z = 9$$
$$2x - y + 3z = -2$$
$$3x - 3y - 4z = 1$$

7. Solve: $$\begin{aligned} x - 8y + z &= 4 \\ 3x - y + 2z &= -1 \end{aligned}$$

8. Solve: $$\begin{aligned} 3x + y + 3z &= 1 \\ x + 2y - z &= 2 \\ 2x - y + 4z &= 4 \end{aligned}$$

9. Solve: $$\begin{aligned} x - y + 2z + w &= 4 \\ y + z - 3 &= 0 \\ z &= w + 2 \end{aligned}$$

10. The perimeter of a triangle is 33 cm. The longest side is 3 cm longer than the medium side. The medium side is twice as long as the shortest side.

    a. If x = length of the shortest side, y = length of the medium side, and z = length of the longest side of the triangle, set up a system of equations that can be used to solve the problem. *(Hint: How many equations will be needed to solve the problem?)*

    b. Find the lengths of the sides.

11. To get the necessary funds for a planned expansion, a small company took out three loans totaling $25,000 for one year. The company was able to borrow some of the money at 8% interest. It borrowed $2000 more than one-half the amount of the 8% loan at 10%. The rest was borrowed at 9%. The total annual interest was $2220. Set up a system of equations that could be used to find the amounts borrowed at each rate. (Remember, interest = (amount borrowed)(annual interest rate as a decimal) For example, $100 borrowed at 3% interest for 1 year costs ($100)(.03) = $3.00 in interest.)

12. Assume problem 10 has a unique solution. If the condition is dropped that the amount borrowed at 10% is $2000 more than ½ the amount borrowed at 8%, how would this affect the system? Would you expect to find a unique solution, many solutions, or no solution? Explain.

13. Suppose the company in problem 10 can borrow only \$6000 at 9%. How would this affect the system of equations? Would you expect to find a unique solution, many solutions, or no solution? Explain.

14. In a physics lab experiment, students are told that a particle moving in a straight line (but not a constant speed!) moves according to the formula $s = at^2 + bt + c$ where s is the particle's distance in feet from a fixed point at time $t$ seconds after launch. Students gather the following data:

| *t* | s |
|---|---|
| 0 | 5 |
| 1 | 23 |
| 2 | 37 |

Find the values for $a$, $b$, and $c$ in the above equation for s. Then use your answer to find how far the particle is from the fixed point 8 seconds after launch.

## Answers:

1. (3, -1) or x=3, y=-1

2. (x, y) = $\left(\frac{2}{3}, \frac{1}{2}\right)$

3a. (4, -1, 0)

3b. no solution

3c. (3-2t, t+5, t)

3d. (4-2s-3t, s, t)

3e. no solution

3f. (2, 3, t)

4. (3, 2, -4)

5. (2, 3, -1)

6. $\left(\frac{-12-15t}{23}, \frac{-13+t}{23}, t)\right)$

7. no solution

8. (1-4t, 1-t, t+2, t)

9. a.
$$x + y + z = 33$$
$$z = y + 3$$
$$y = 2x$$

9b. x=6 cm
y = 12 cm
z = 15 cm

10.
$$x + y + z = 25{,}000$$
$$.08x + .10y + .09z = 2220$$
$$y = \frac{1}{2}x + 2000$$

or

$$x + y + z = 25{,}000$$
$$8x + 10y + 9z = 222{,}000$$
$$-x + 2y \qquad = 4000$$

11. would become 2 equations with 3 variables which is a dependent system. Many solutions.

12. would be 4 equations with 3 variables (or 3 equations with 2 variables). No solution.

15. a. a = -2, b = 20, c = 5, so $s = -2t^2 + 20t + 5$

b. when t = 8, s = 37 feet

**Detailed Solutions** (start on next page)

1. a. Gaussian elimination on $x+2y=1$
$2x-y=7$

The initial augmented coefficient matrix is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 7 \end{array}\right] = A$

Remember, the desired final matrix is $\left[\begin{array}{cc|c} 1 & 0 & \text{(number)} \\ 0 & 1 & \text{(number)} \end{array}\right]$

The allowable row operations are:

1. Rows may be interchanged. (This would be the same as writing the original equations is a different order.)
2. A row can be multiplied by a constant. (This would be the same as multiplying both sides of an equation by a constant.)
3. A row can be replaced by that row (or a multiple of that row) + or − a multiple of any other row. (This is the same as, in non-matrix elimination solutions, adding a multiple of one equation to a multiple of another equation.)

*note: to refer to specific entries in a matrix, it is convenient to give them names, using subscripts to describe the location. The first subscript is the row, the second is the column where the entry is located. In our initial matrix A, "7" is $a_{23}$ – the entry in row 2, column 3.

Steps: $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 7 \end{array}\right]$ Since there is already a "1" in the top left position, nothing needs to be done to the first row now.

To get a "0" in the $a_{21}$ position (see note*), add (−2)*row 1 to row 2 and store the result in row 2 (replacing the original row 2).

$\xrightarrow{R_2-2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & 5 \end{array}\right]$

$R_2$: 2 −1 7
$-2R_1$: −2 −4 −2
new $R_2$: 0 −5 5

Next, get a "1" in the $a_{22}$ position. This is almost always done by dividing the row by the $a_{22}$ number.

$\xrightarrow{R_2/(-5) \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$

This is the final matrix for the Gaussian elimination method.

Note the "triangular" shape of the non-zero entries, (not including the augmented last column).

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

This matrix corresponds to the equations.

$$1x + 2y = 1$$
$$0x + 1y = -1$$

or

$$x + 2y = 1$$
$$\boxed{y = -1}$$

. Back-substituting into the first equation,

$$x + 2y = 1$$
$$x + 2(-1) = 1$$
$$x - 2 = 1 \implies \boxed{x = 3}$$

The solution is $\boxed{(x, y) = (3, -1)}$ (as an ordered pair).

Note: although the video on 2x2 systems states that the variables should be in alphabetical order, that is not necessary. The variables should, however, be in some known order, and be in the same order in each equation.

b. To complete this problem using Gauss-Jordan elimination, start with the final matrix from the Gaussian elimination process:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

Continue by making $a_{12} = 0$ by adding $(-2)(\text{row } 2)$ to row 1.

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{rccc} R_1: & 1 & 2 & 1 \\ -2R_2: & 0 & -2 & 2 \\ \hline & 1 & 0 & 3 \end{array}$$

Now the solution can be read immediately, since the equations associated with this matrix are

$$\left.\begin{array}{l} 1x + 0y = 3 \\ 0x + 1y = -1 \end{array}\right\} \text{or} \to \begin{array}{l} x = 3 \\ y = -1 \end{array}$$

$$(x, y) = \boxed{(3, -1)}$$

2. $3x + 4y = 4$
$6x - 2y = 3$ $\longrightarrow \left[\begin{array}{cc|c} 3 & 4 & 4 \\ 6 & -2 & 3 \end{array}\right]$

To get $a_{11} = 1$, divide row 1 by 3: $\xrightarrow{R_1/3 \to R_1} \left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 6 & -2 & 3 \end{array}\right]$

Remember the "addresses" of the matrix entries.

$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]$

Notes: Most errors is solving are due to arithmetic mistakes. Help avoid these by
1. enter the rows NOT being changed first.
2. do the arithmetic off to the side instead of in your head to get better accuracy.

These are especially important as the calculations get more involved (like with fractions) and the systems get larger.

Now we need "0" in $a_{21}$ position: $\left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 6 & -2 & 3 \end{array}\right] \xrightarrow{R_2 - 6R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 0 & -10 & -5 \end{array}\right]$

$R_2$: $6 \quad -2 \quad 3$
$-6R_1$: $-6 \quad -6\left(\frac{4}{3}\right) \quad -6\left(\frac{4}{3}\right)$

$= \begin{array}{ccc} 6 & -2 & 3 \\ -6 & -8 & -8 \\ \hline 0 & -10 & -5 \end{array}$ ← new $R_2$

Need "1" in $a_{22}$: $\left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 0 & -10 & -5 \end{array}\right] \xrightarrow{-\frac{1}{10}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 0 & 1 & \frac{-5}{-10} = \frac{1}{2} \end{array}\right]$

Need "0" in $a_{12}$: $\left[\begin{array}{cc|c} 1 & 4/3 & 4/3 \\ 0 & 1 & 1/2 \end{array}\right] \xrightarrow{R_1 - \frac{4}{3}R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/2 \end{array}\right]$ ← new $R_1$

$R_1$: $1 \quad \frac{4}{3} \quad \frac{4}{3}$
$-\frac{4}{3}R_2$: $0 \quad -\frac{4}{3} \quad -\frac{4}{3}\left(\frac{1}{2}\right) = -\frac{2}{3}$

$= \begin{array}{ccc} 1 & 4/3 & 4/3 \\ 0 & -4/3 & -2/3 \\ \hline 1 & 0 & 2/3 \end{array}$

check:
$3(2/3) + 4\left(\frac{1}{2}\right) = 2 + 2 = 4$ ✓
$6\left(\frac{2}{3}\right) - 2\left(\frac{1}{2}\right) = 4 - 1 = 3$ ✓

Solution: $(x, y) = \boxed{(2/3,\ 1/2)}$

or $\boxed{x = \frac{2}{3},\ y = \frac{1}{2}}$

3. a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ solution: $\boxed{(4,-1,0)}$ or $x=4$, $y=-1$, $z=0$ (consistent, independent equations)

b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\boxed{\text{No solution}}$ because row 3 says $0=1$ which is a contradiction. (inconsistent equations)

c. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 5 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$ ← Row 3 can be ignored. It only tells us $0=0$.

Row 2: $y-z=5$ if we let $\boxed{z=t}$ where $t$ can be any real number. ($t$ is called a "parameter")

then $y=z+5$

$\boxed{y=t+5}$

Row 1: $x+2z=3 \rightarrow x=3-2z$

$\boxed{x=3-2t}$

Solution: $\boxed{(3-2t,\ t+5,\ t)}$ (Infinite number of solutions, but all of this form.)

one possible solution: if $t=0$ $x=3, y=5, z=0$ $(3,5,0)$

if $t=-1$ $x=5, y=4$ $z=-1$ $(5,4,-1)$

dependent system: (one unique solution for each value of $t$.)

d. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \rightarrow x+2y+3z=4$

In this case, we have a free choice of values for <u>two</u> of the variables before the third variable is uniquely determined. So we need <u>two</u> <u>parameters</u>.

let $\boxed{z=t}$ and $\boxed{y=s}$ then $\boxed{x=4-2s-3t}$

solution: $(4-2s-3t,\ s,\ t)$ where $s,t$ can be any real numbers.

for example, if $s=3$ and $t=-2$,

$x=4-2(3)-3(-2)=4$

$y=s=3$

$z=t=-2 \rightarrow$ soln. $(4,3,-2)$

dependent system: (one unique solution for each value of $s$ and $t$)

3.e. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\boxed{\text{No Solution!}}$ (inconsistent system)

It starts out looking like $x=2$ and $y=3$ but it means
$x=2$, $y=3$ when $0=1$
(which is never!).

f. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$

Now, this is a dependent system of equations, but the equations say:

$x=2$
$y=3$
$z$ (no information!)

Solution: $(2,3,t)$ where $t=$ any real number

4. $\begin{aligned} -x+y &= -1 \\ y-z &= 6 \\ x+z &= -1 \end{aligned} \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 6 \\ 1 & 0 & 1 & -1 \end{array}\right]$ ← Note: we usually get the "1" on the diagonal by dividing the row by the value in the diagonal position, but here it was as easy to interchange rows.

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 6 \\ 1 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_1]{R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ -1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R_3+R_1 \to R_3 \,^*} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-R_2 \to R_3 \,^*} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 2 & -8 \end{array}\right] \xrightarrow{\frac{R_3}{2} \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow{R_2+R_3 \to R_2 \,^*} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

(at this point, we know $z=-4$. Gaussian elimination would stop here, then finish using back-substitution.)

$$\xrightarrow{R_1-R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

Solution: $\boxed{\begin{aligned} x &= 3 \\ y &= 2 \\ z &= -4 \end{aligned}}$ or $\boxed{(3,2,-4)}$
(ordered triple)

* usually, we get a "0" entry by adding a multiple of the row that contains the "1" on the diagonal in that column. This prevents "undoing" work done previously.

5. $x+2y-z=9$
$2x-y+3z=-2$
$3x-3y-4z=1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & -1 & 3 & -2 \\ 3 & -3 & -4 & 1 \end{array}\right] \xrightarrow[R_3-3R_1 \to R_3]{R_2-2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -5 & 5 & -20 \\ 0 & -9 & -1 & -26 \end{array}\right] \xrightarrow{\frac{R_2}{-5} \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & 1 & -1 & 4 \\ 0 & -9 & -1 & -26 \end{array}\right]$$

column 1 is done.

can do 2 steps at once if careful!

$$\begin{array}{rrrrrl} R_2: & 2 & -1 & 3 & -2 & \\ -2R_1: & -2 & -4 & 2 & -18 & \\ \hline & 0 & -5 & 5 & -20 & \to R_2 \end{array}$$

$$\begin{array}{rrrrrl} R_3: & 3 & -3 & -4 & 1 & \\ -3R_1: & -3 & -6 & 3 & -27 & \\ \hline & 0 & -9 & -1 & -26 & \to R_3 \end{array}$$

$$\begin{array}{rrrrrl} R_1: & 1 & 2 & -1 & 9 & \\ -2R_2: & 0 & -2 & 2 & -8 & \\ \hline & 1 & 0 & 1 & 1 & \to R_1 \end{array}$$

$$\begin{array}{rrrrrl} R_3: & 0 & -9 & -1 & -26 & \\ +9R_2: & 0 & 9 & -9 & 36 & \\ \hline & 0 & 0 & -10 & 10 & \to R_3 \end{array}$$

$$\xrightarrow[R_3+9R_2 \to R_3]{R_1-2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -10 & 10 \end{array}\right] \xrightarrow{\frac{R_3}{-10} \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

columns 1 & 2 are done.

$$\xrightarrow[R_2+R_3 \to R_2]{R_1-R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Solution: $\boxed{x=2,\ y=3,\ z=-1}$

or $(x,y,z)=\boxed{(2,3,-1)}$

(consistent, independent system)

6. $x-8y+z=4$
$3x-y+2z=-1$

note: since there are more variables than equations, we expect a dependent system (infinite solutions).

$$\left[\begin{array}{ccc|c} 1 & -8 & 1 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right] \xrightarrow{R_2-3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -8 & 1 & 4 \\ 0 & 23 & -1 & -13 \end{array}\right] \xrightarrow{\frac{R_2}{23} \to R_2} \left[\begin{array}{ccc|c} 1 & -8 & 1 & 4 \\ 0 & 1 & -\frac{1}{23} & -\frac{13}{23} \end{array}\right]$$

$$\begin{array}{rrrrr} R_2: & 3 & -1 & 2 & -1 \\ -3R_1: & -3 & 24 & -3 & -12 \\ \hline & 0 & 23 & -1 & -13 \end{array}$$

let $\boxed{z=t}$

Row 2: $y-\frac{1}{23}z=\frac{-13}{23}$

$\boxed{y=\frac{-13}{23}+\frac{1}{23}t}$

Row 1: $x-8\left(\frac{-13}{23}+\frac{1}{23}t\right)+t=4$

$x+\frac{104}{23}-\frac{8}{23}t+t=4$

$x+\frac{15}{23}t=4\left(\frac{23}{23}\right)-\frac{104}{23} \Rightarrow \boxed{x=\frac{-12}{23}-\frac{15}{23}t}$

Solution: $\left(\frac{-12-15t}{23},\ \frac{-13+t}{23},\ t\right)$

checking answer to #6: $x - 8y + z = 4$

$$\frac{-12-15t}{23} - 8\left(\frac{-13+t}{23}\right) + t \stackrel{?}{=} 4$$

$$-\frac{12}{23} - \frac{15}{23}t + \frac{104}{23} - \frac{8}{23}t + \frac{23}{23}t =$$

$$-\frac{12}{13} + \frac{104}{23} - \frac{15}{23}t - \frac{8}{23}t + \frac{23}{23}t =$$

$$\frac{92}{23} - \cancel{\frac{23}{23}t} + \cancel{\frac{23}{23}t} = \frac{92}{23} = 4 \quad \text{check!}$$

$$3x - y + 2z = 3\left(-\frac{12}{23} - \frac{15}{23}t\right) - \left(\frac{-13+t}{23}\right) + 2(t) \stackrel{?}{=} -1$$

$$= -\frac{36}{23} - \frac{45}{23}t + \frac{13}{23} - \frac{1}{23}t + \frac{46}{23}t$$

$$= \frac{-36}{23} + \frac{13}{23} - \frac{45}{23}t - \frac{1}{23}t + \frac{46}{23}t$$

$$= -\frac{23}{23} - \cancel{\frac{46}{23}t} + \cancel{\frac{46}{23}t} = -\frac{23}{23} = -1 \quad \text{check!}$$

7. $$\begin{aligned} 3x + y + 3z &= 1 \\ x + 2y - z &= 2 \\ 2x - y + 4z &= 4 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & 3 & 1 \\ 1 & 2 & -1 & 2 \\ 2 & -1 & 4 & 4 \end{array}\right] \xrightarrow[R_2 \to R_1]{R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 3 & 1 & 3 & 1 \\ 2 & -1 & 4 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -5 & 6 & -5 \\ 0 & -5 & 6 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -5 & 6 & -5 \\ \underbrace{0 \quad 0 \quad 0}_{} & & & 5 \end{array}\right]$$

inconsistent

$0 = 5$ is a contradiction.

$\boxed{\text{no solution}}$

$$\begin{array}{rrrrr} R_2: & 3 & 1 & 3 & 1 \\ -3R_1: & -3 & -6 & 3 & -6 \\ \hline & 0 & -5 & 6 & -5 \end{array} \qquad \begin{array}{rrrrr} R_3: & 2 & -1 & 4 & 4 \\ -2R_1: & -2 & -4 & 2 & -4 \\ \hline & 0 & -5 & 6 & 0 \end{array}$$

8. $x - y + 2z + w = 4$
$y + z - 3 = 0$
$z = w + 2$

This problem might be faster to solve just using substitution.

$z = w + 2$

$y = 3 - z = 3 - (w+2) = 1 - w$.

then $x - (1-w) + 2(w+2) + w = 4$

$x - 1 + w + 2w + 4 + w = 4$

$x + 3 + 4w = 4$

$x = -4w + 1$

soln: $(x, y, z, w) = (-4w+1, 1-w, w+2, w)$
where $w$ is any real number.

but using Gauss-Jordan elimination . . .

Row 2: $y + z = 3$
Row 3: $z - w = 2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

Column 1 is already in final form.

To finalize column 2, eliminate the $(-1)$ in $a_{12}$ position.

$$\xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 7 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

To finalize column 3, add multiples of row 3 to rows 1 and 2

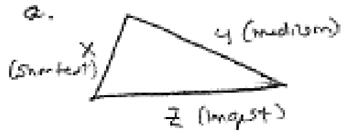
$$\xrightarrow{R_1 - 3R_3 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x + 4w = 1 \\ y + w = 1 \\ z - w = 2 \end{array}$$

if we let $\boxed{w = t}$, then $\boxed{\begin{array}{l} x = 1 - 4t \\ y = 1 - t \\ z = t + 2 \end{array}}$

Solution: $(x, y, z, w) = \boxed{(1-4t, 1-t, t+2, t)}$

(same as regular substitution method)

9. a.




perimeter = 33 $\Rightarrow$ $x + y + z = 33$

longest side is 3 longer than medium side $\Rightarrow$ $z = y + 3$

medium side is twice shortest side $\Rightarrow$ $y = 2x$

system:
$$\left.\begin{aligned} x + y + z &= 33 \\ z &= y + 3 \\ y &= 2x \end{aligned}\right\}$$
3 variables – need 3 eqns. to solve.

b. rearranging:
$$\begin{aligned} x + y + z &= 33 \\ -y + z &= 3 \\ -2x + y &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 33 \\ 0 & -1 & 1 & 3 \\ -2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 + 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 33 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 2 & 66 \end{array}\right] \xrightarrow{(-1)R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 33 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 2 & 66 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_2 \to R_3]{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 36 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 5 & 75 \end{array}\right] \xrightarrow{\frac{R_3}{5} \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 36 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 15 \end{array}\right] \xrightarrow[R_1 - 2R_3 \to R_1]{R_2 + R_3 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 15 \end{array}\right] \to \boxed{\begin{aligned} x &= 6 \text{ cm} \\ y &= 12 \text{ cm} \\ z &= 15 \text{ cm.} \end{aligned}}$$

10. let $x$ = amt. borrowed at 8%
$y$ = amt. borrowed at 10%
$z$ = amt. borrowed at 9%

total borrowed = \$25,000 $\rightarrow$ $x + y + z = 25{,}000$

total interest = \$2220 $\rightarrow$ $.08x + .10y + .09z = 2220$

or $8x + 10y + 9z = 222{,}000$

borrowed \$2000 more than $\frac{1}{2}$ amt. at 8% at 10% $\rightarrow$ $y = \frac{1}{2}x + 2000$

$2y = x + 4000$

$-x + 2y = 4000$

eqns: $\boxed{\begin{array}{c} x + y + z = 25{,}000 \\ .08x + .10y + .09z = 2220 \\ y = \frac{1}{2}x + 2000 \end{array}}$ or $\boxed{\begin{array}{l} x + y + z = 25{,}000 \\ 8x + 10y + 9z = 222{,}000 \\ -x + 2y \quad = 4{,}000 \end{array}}$

11. If equations in 10 have a unique solution and if the condition that $y = \frac{1}{2}x + 2000$ is dropped, there would only be 2 equations with 3 variables. $\rightarrow$ dependent system.

There would be $\boxed{\text{an infinite number of solutions.}}$

(with the real-world restriction that $x$, $y$, and $z$ would all have to be positive, that would limit the value of the parameter.)

12. If the restriction that $z = 6{,}000$, the system becomes.

$x + y + 6000 = 25{,}000 \rightarrow x + y = 19{,}000$

$8x + 10y + 9(6000) = 222{,}000 \rightarrow 8x + 10y = 168{,}000$

$-x + 2y = 4{,}000$

Now we would expect no solution (inconsistent system) since there are more equations than variables.

13. a. $s = at^2 + bt + c$

if $t=0$, $s=5$ $\longrightarrow$ $s = a(0) + b(0) + c = 5$

$c = 5$ ← make this equation 3

if $t=1$, $s=23$ $\rightarrow$ $s = a(1)^2 + b(1) + c = 23$

$a + b + c = 23$ ← eqn. 1

if $t=2$, $s=37$ $\rightarrow$ $s = a(2)^2 + b(2) + c = 37$

$4a + 2b + c = 37$ ← eqn. 2

in matrix form: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 23 \\ 4 & 2 & 1 & 37 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$\xrightarrow{R_2 - 4R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 23 \\ 0 & -2 & -3 & -55 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{\frac{R_2}{-2} \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 23 \\ 0 & 1 & 3/2 & 55/2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$$\begin{array}{rrrrr} R_2: & 4 & 2 & 1 & 37 \\ -4R_1: & -4 & -4 & -4 & -92 \\ \hline & 0 & -2 & -3 & -55 \end{array}$$

$\xrightarrow{R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & -9/2 \\ 0 & 1 & 3/2 & 55/2 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow[R_1 + \frac{1}{2}R_3 \to R_1]{R_2 - \frac{3}{2}R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 5 \end{array}\right]$

$$\begin{array}{rrrrl} R_1: & 1 & 1 & 1 & 23 = \frac{46}{2} \\ -R_2 & 0 & -1 & -3/2 & -55/2 \\ \hline & 1 & 0 & -1/2 & -9/2 \end{array}$$

$$\begin{array}{rrrrl} R_2: & 0 & 1 & \frac{3}{2} & \frac{55}{2} \\ -\frac{3}{2}R_3 & 0 & 0 & -\frac{3}{2} & -\frac{15}{2} \\ \hline & 0 & 1 & 0 & \frac{40}{2} = 20 \end{array}$$

$$\begin{array}{rrrrl} R_1: & 1 & 0 & -\frac{1}{2} & -\frac{9}{2} \\ +\frac{1}{2}R_3: & 0 & 0 & \frac{1}{2} & \frac{5}{2} \\ \hline & 1 & 0 & 0 & -\frac{4}{2} = -2 \end{array}$$

solution: $\boxed{a = -2 \quad b = 20 \quad c = 5}$

equation of motion: $\boxed{s = -2t^2 + 20t + 5}$

b. 8 seconds after launch, $s(8) = -2(8)^2 + 20(8) + 5$
$= -128 + 160 + 5 = \boxed{37 \text{ feet}}$

**Additional Practice:**

1. Go To http://kutasoftware.com/freeipc.html

2. **Under "Matrices and Systems":**
   - Multivariable linear systems and row operations

3. Go to http://www.kutasoftware.com/freeia2.html

4. **Under "Systems of Equations and Inequalities":**
   - These systems of equations were intended to be solved by other methods but answers are given and any can be solved using Gauss-Jordan elimination.

5. You can print out the worksheets and work on them. The solutions are provided at the end of the worksheets

6. For help, please contact the ***Math Assistance Area*.**